Tutorial - 8

1. Examine the impact of channel non-linearity on FM modulation, i.e., let

$$s(t) = A_c \cos(2\pi f_c t + \phi(t))$$

where $\phi(t) = 2\pi k_f \int_0^t m(x)\,dx$ is the phase of $s(t)$, be the FM signal.

Let the output of the channel be given by

$$v(t) = a_1 s(t) + a_2 s^2(t) + a_3 s^3(t)$$

Analyse the spectrum of $v(t)$ and show how $s(t)$ can be recovered from $v(t)$ using a band pass filter.

What would have happened if $s(t)$ was an AM signal? Which modulation scheme (AM or FM) is more robust to channel non-linearity?

Solution 1. $s(t) = A_c \cos(2\pi f_c t + \phi(t))$, (2)

$$s^2(t) = A_c^2 \cos^2(2\pi f_c t + \phi(t))$$

$$= \frac{A_c^2}{2} + \frac{A_c^2}{2}\cos(4\pi f_c t + 2\phi(t)),$$

and

$$s^3(t) = A_c^3 \cos^3(2\pi f_c t + \phi(t))$$

using the identity.

$$\cos^3\theta = \frac{\cos 3\theta + 3\cos\theta}{4}$$, we get.

$$s^3(t) = \frac{3A_c^3}{4}\cos(2\pi f_c t + \phi(t)) + \frac{A_c^3}{4}\cos(6\pi f_c t + 3\phi(t)).$$

$$\therefore \quad v(t) = a_1 s(t) + a_2 s^2(t) + a_3 s^3(t)$$

$$= a_1 A_c \cos(2\pi f_c t + \phi(t))$$

$$+ \frac{a_2 A_c^2}{2} + \frac{a_2 A_c^2}{2}\cos(4\pi f_c t + 2\phi(t))$$

$$+ \frac{3a_3 A_c^3}{4}\cos(2\pi f_c t + \phi(t))$$

$$+ \frac{a_3 A_c^3}{4}\cos(6\pi f_c t + 3\phi(t))$$

(1)

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$$\therefore \hat{v}(t) = \underbrace{\frac{a_2 A_c^2}{2}}_{T_1} + \underbrace{\left(a_1 A_c + \frac{3a_3 A_c^3}{4}\right)\cos(2\pi f_c t + \phi(t))}_{T_2}$$

$$+ \underbrace{\frac{a_2 A_c^2}{2}\cos(4\pi f_c t + 2\phi(t))}_{T_3}$$

$$+ \underbrace{\frac{a_3 A_c^3}{4}\cos(6\pi f_c t + 3\phi(t))}_{T_4}. \quad \text{— (2)}$$

we are interested in this term.

The term T_1 is at D.C.

Using Carson's rule of thumb, T_2 occupies a bandwidth $B_{T_2} \simeq 2\Delta f + 2B_m$

where $\Delta f = k_f \max_t |m(t)|$ and.

B_m = bandwidth $m(t)$ ($m(t)$ is band limited to $[-B_m, B_m]$).

T_2 lies in the band.

$$\left(f_c - \frac{B_{T_2}}{2},\ f_c + \frac{B_{T_2}}{2}\right). \quad \text{— (3)}.$$

similarly T_3 has a bandwidth.

$$B_{T_3} \simeq 4\Delta f + 2B_m$$

since $2\phi(t)$ has a frequency deviation two times the frequency deviation of $\phi(t)$.

So, T_3 is restricted to the band

$$\left(2f_c - \frac{B_{T_3}}{2},\ 2f_c + \frac{B_{T_3}}{2}\right) \longleftarrow (4).$$

Similarly it can be shown that T_4 is restricted to the band

$$\left(3f_c - \frac{B_{T_4}}{2},\ 2f_c + \frac{B_{T_4}}{2}\right) \text{ where}$$

$$B_{T_4} \simeq 6\Delta f + 2B_m.$$

$\therefore$ To separate the term T_2, we would like the band for T_2 to be disjoint from the occupancy bands of the other terms.

This is possible if

$$f_c + \frac{B_{T_2}}{2} < 2f_c - \frac{B_{T_3}}{2} \quad \text{--- (5.1)}$$

and $$f_c - \frac{B_{T_2}}{2} > 0 \quad \text{--- (5.2)}$$

and $$f_c + \frac{B_{T_2}}{2} < 3f_c - \frac{B_{T_4}}{2} \quad \text{--- (5.3)}$$

From condition 5.2 we get:

$$f_c > \Delta f + B_m \quad \text{—— } 6.1$$

and from 5.3 we get:

$$2f_c > 4\Delta f + 2B_m$$

$$\text{i.e. } f_c > 2\Delta f + B_m. \quad \text{—— } 6.2$$

From 5.1 we get:

$$f_c > 3\Delta f + B_m. \quad \text{—— } 6.3$$

From 6.1 – 6.3 it follows that

∴ If $f_c > 3\Delta f + B_m$, then

it is true that the occupancy band of T_2 is disjoint from the occupancy bands of the other terms,

and so by putting a band pass filter with response

$$B(f) = \begin{cases} 1, & |f - f_c| < \frac{B_{T_2}}{2} \\ 1, & |f + f_c| < \frac{B_{T_2}}{2} \\ 0, & \text{otherwise} \end{cases}$$

we can recover

the term

$$T_2 = \left(a_1 A_c + \frac{3 a_3 A_c^3}{4}\right) \cos\left(2\pi f_c t + \phi(t)\right).$$

~~this~~ compared to

$$s(t) = A_c \cos\left(2\pi f_c t + \phi(t)\right),$$

the amplitude of T_2 is distorted, which however is not a matter of concern since the ~~message~~ ~~is~~ ~~signal~~ information is carried by $\phi(t)$ which is the same in T_2 and $s(t)$.

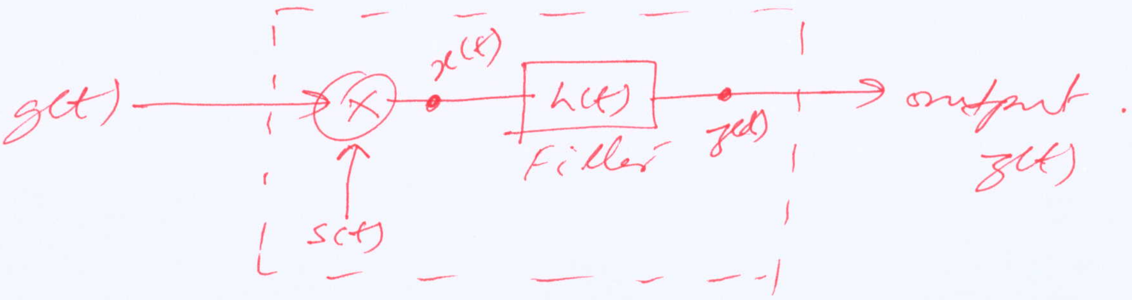
In comparison, if $s(t)$ was an AM signal, it is clear that due to channel non-linearity the amplitude would get distorted, and since the amplitude carries information, there would be no way of recovering back $m(t)$ from the output of the non-linear channel. This shows that a constant envelope FM signal (where the phase carries information) is more robust to channel non-linearities when compared to an AM signal (where the amplitude carries information).

4.12. Real-time spectrum analyzer.

We would like to ~~analyze the~~ view the frequency spectrum (amplitude) of a baseband signal $g(t)$.

i.e., we would like to ~~have a device~~ design a device whose output is $|G(f)|$.

Consider the device



where $s(t) = \cos(2\pi f_c t - \pi k t^2)$, and

$h(t) = \cos(2\pi f_c t + \pi k t^2)$.

We would like to analyze the envelope of the output $z(t)$, i.e.,

since $z(t)$ is a passband signal we will have

$$z(t) = z^I(t)\cos 2\pi f_c t - z^Q(t)\sin 2\pi f_c t$$

whose envelope would be $\sqrt{z^{I^2}(t) + z^{Q^2}(t)}$

$$= |z^I(t) + j z^Q(t)| = |\tilde{z}(t)|$$

where $\tilde{y}(t)$ is the complex baseband representation of $y(t)$.

We have

$$x(t) \longrightarrow \boxed{\begin{array}{c}\text{filter} \\ h(t)\end{array}} \longrightarrow y(t)$$

$\therefore$ We know that filtering in passband is equivalent to filtering in complex baseband and therefore

$$\tilde{x}(t) \longrightarrow \boxed{\frac{\tilde{h}(t)}{2}} \longrightarrow \tilde{y}(t)$$

where $\tilde{x}(t)$ and $\tilde{h}(t)$ are the complex baseband representation of $x(t)$ and $h(t)$ respectively.

Note that
$$\begin{aligned} h(t) &= \cos(2\pi f_c t + \pi k t^2) \\ &= Re\left(e^{j2\pi f_c t} \cdot e^{j\pi k t^2}\right) \\ &= Re\left(\tilde{h}(t)\, e^{j2\pi f_c t}\right) \end{aligned}$$

$$\therefore \tilde{h}(t) = e^{j\pi k t^2}.$$

Further since
$$\begin{aligned} x(t) &= g(t)\, s(t). \\ &= Re\left(\tilde{s}(t) e^{j2\pi f_c t}\right) g(t) \\ &= Re\left(g(t)\, \tilde{s}(t) e^{j2\pi f_c t}\right) \end{aligned}$$

since $g(t)$ is baseband

$\therefore x(t) = Re\left(g(t)\,\tilde{s}(t)\,e^{j2\pi f_c t}\right)$

$= Re\left(\tilde{x}(t)\,e^{j2\pi f_c t}\right)$

from where we get

$$\tilde{x}(t) = g(t)\,\tilde{s}(t)$$

since $s(t) = \cos(2\pi f_c t - \pi k t^2)$

$= Re\left(e^{j2\pi f_c t} \cdot e^{-j\pi k t^2}\right)$

$= Re\left(e^{j2\pi f_c t} \cdot \tilde{s}(t)\right)$

we have

$$\tilde{s}(t) = e^{-j\pi k t^2}.$$

$\therefore \tilde{y}(t) = \frac{1}{2}\,\tilde{h}(t) \circledast \tilde{x}(t)$

$= \frac{1}{2}\left(\left(g(t)\,e^{-j\pi k t^2}\right) \circledast e^{j\pi k t^2}\right)$

$= \frac{1}{2}\int_{-\infty}^{\infty} g(\tau)\,e^{-j\pi k \tau^2}\,e^{j\pi k (t-\tau)^2}\,d\tau.$

$= \frac{1}{2}\int_{-\infty}^{\infty} g(\tau)\,e^{j\pi k t^2}\,e^{-j\pi k t 2\tau}\,d\tau.$

$= \frac{e^{j\pi k t^2}}{2}\int_{-\infty}^{\infty} g(\tau)\,e^{-j2\pi k t \tau}\,d\tau.$

$\therefore$

$$z^I(t) + j z^Q(t) = \tilde{z}(t)$$

$$= \frac{e^{j\pi k t}}{2} \int_{-\infty}^{\infty} g(\tau)\, e^{-j2\pi k t \tau}\, d\tau .$$

envelope of $z(t)$ is $\sqrt{z^{I^2}(t) + z^{Q^2}(t)}$

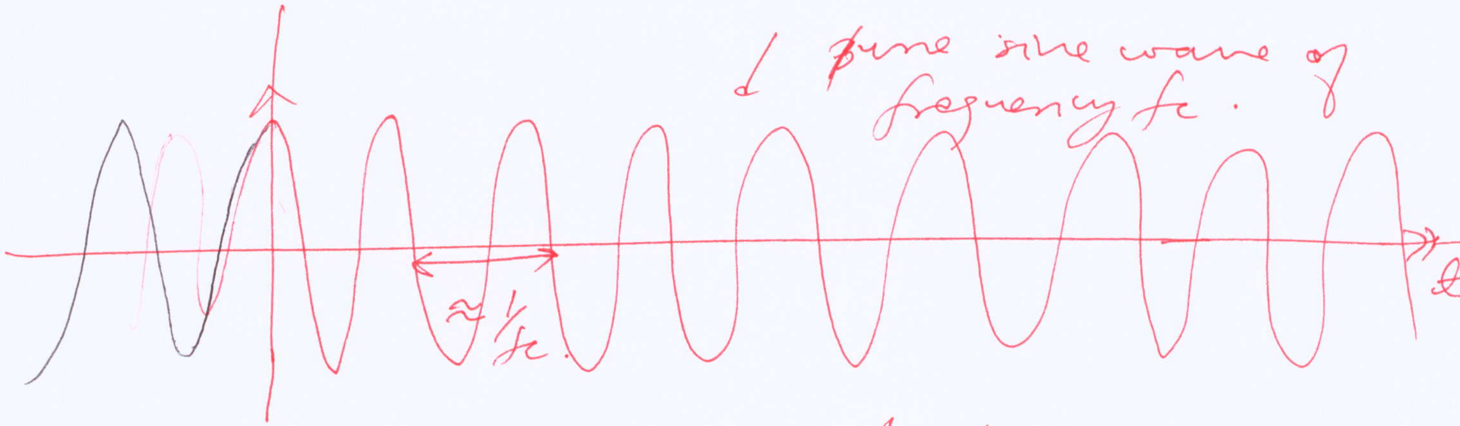
(what is the practical significance of the envelope of a passband signal which has a small bandwidth compared to its center frequency ??)

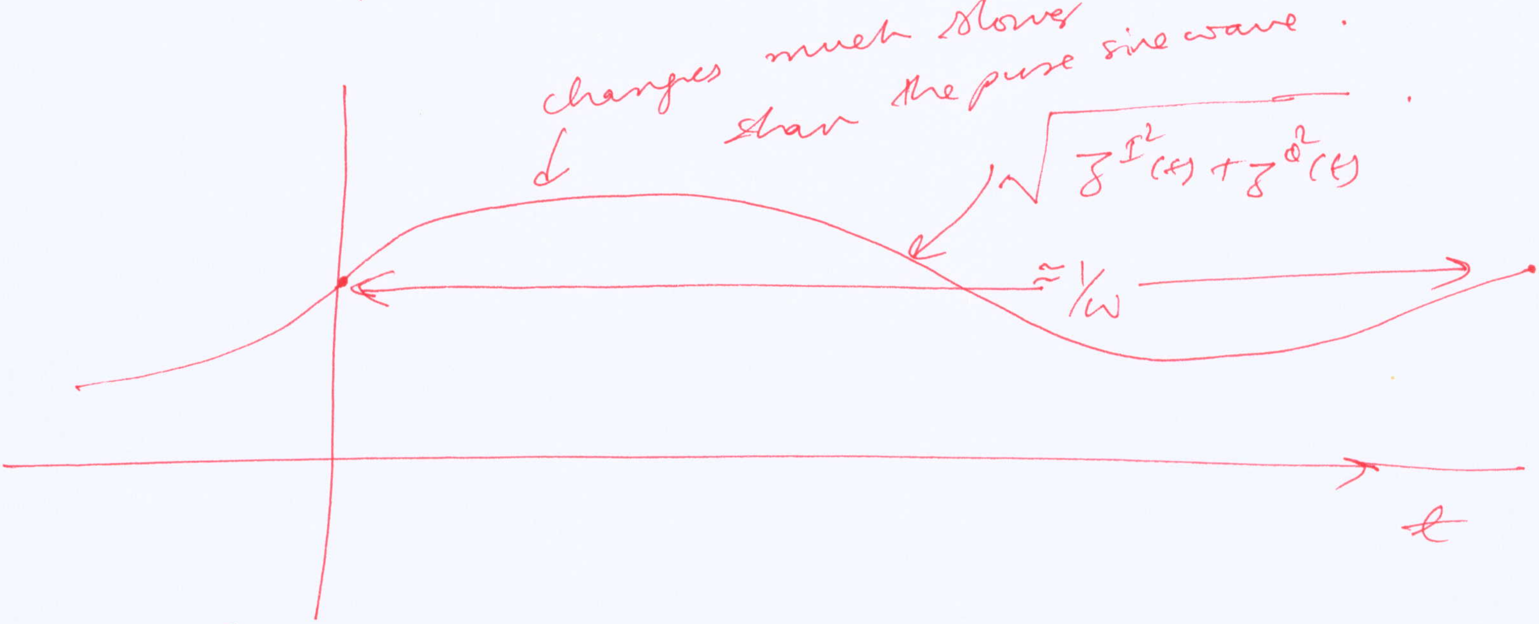
$$z(t) = z^I(t)\cos 2\pi f_c t - z^Q(t)\sin 2\pi f_c t$$

$$= \sqrt{z^{I^2}(t) + z^{Q^2}(t)} \left\{ \cos 2\pi f_c t \frac{z^I(t)}{\sqrt{z^{I^2}(t) + z^{Q^2}(t)}} - \sin 2\pi f_c t \frac{z^Q(t)}{\sqrt{z^{I^2}(t) + z^{Q^2}(t)}} \right\}$$

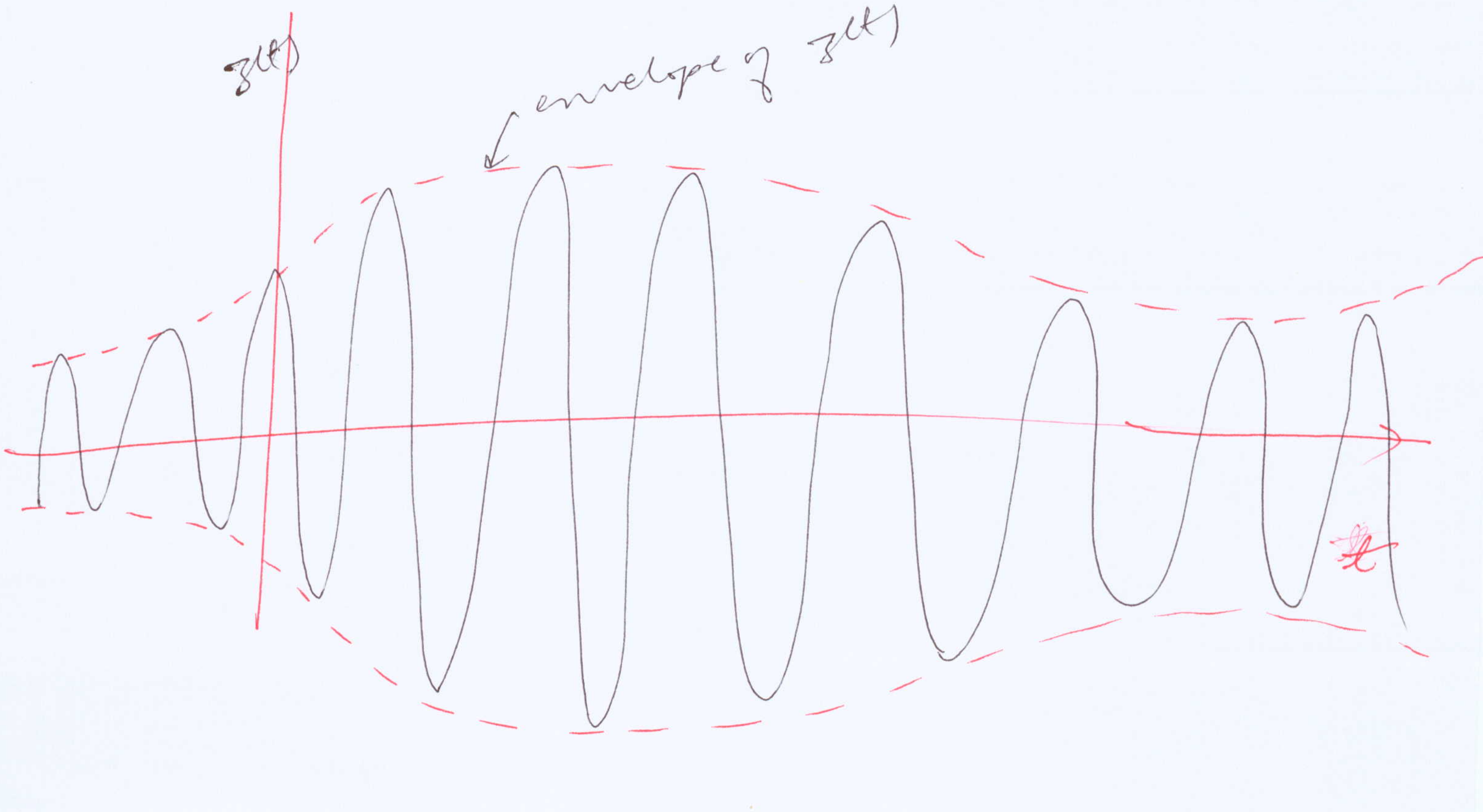
$$= \sqrt{z^{I^2}(t) + z^{Q^2}(t)} \cos\left(2\pi f_c t + \tan^{-1}\left(\frac{z^Q(t)}{z^I(t)}\right)\right)$$

$\therefore$ Since $z^I(t)$ and $z^Q(t)$ are baseband and have a bandwidth much smaller than f_c, $z(t)$ appears like a pure sine wave of frequency f_c multiplied by $\sqrt{z^{I^2}(t) + z^{Q^2}(t)}$.





The product of the above two signals.



$\therefore$ envelope of $g(t) = |\tilde{g}(t)|$

$$= \left| \frac{e^{j\pi k t^2}}{2} \int_{-\infty}^{\infty} g(\tau) e^{-j2\pi k t \tau} d\tau \right|$$

$$= \frac{1}{2} \left| \int_{-\infty}^{\infty} g(\tau) e^{-j2\pi k t \tau} d\tau \right|$$

$$= \frac{1}{2} \left| G(f = kt) \right|$$

where $G(f) \triangleq \int_{-\infty}^{\infty} g(\tau) e^{-j2\pi f \tau} d\tau$

is the Fourier transform of $g(t)$.

$\therefore$ At time t, the spectrum analyzer outputs $\frac{|G(f=kt)|}{2}$, i.e., the amplitude spectrum of g at $f = kt$, i.e., the spectrum analyzer linearly sweeps the frequency ~~domain~~ spectrum at the rate of k Hertz/second.