2.9 a)

Let $h(t) = g_1(t) * g_2(t)$

$$\frac{dh(t)}{dt} \rightleftarrows j2\pi f H(f)$$
$$= j2\pi f G_1(f) G_2(f)$$
$$= \left(j2\pi f G_1(f) \right) G_2(f)$$

$$\left(j2\pi f G_1(f) \right) G_2(f) \rightleftarrows \left[\frac{dg_1(t)}{dt} \right] * g_2(t)$$

$$\therefore \frac{d}{dt} \left[g_1(t) * g_2(t) \right] = \left[\frac{dg_1(t)}{dt} \right] * g_2(t)$$

b)
$$\int_{-\infty}^{t} g_1(t) * g_2(t) dt \rightleftarrows \frac{1}{j2\pi f} G_1(f) G_2(f) + \frac{G_1(0) G_2(0)}{2} \delta(f)$$
$$= \left[\frac{1}{j2\pi f} G_1(f) \right] G_2(f) + \left[\frac{G_1(0)}{2} \delta(f) \right] G_2(f)$$
$$= \left[\frac{1}{j2\pi f} G_1(f) + \frac{G_1(0)}{2} \delta(f) \right] G_2(f)$$

$$\therefore \int_{-\infty}^{t} g_1(t) * g_2(t) dt = \left[\int_{-\infty}^{t} g_1(t) \right] * g_2(t)$$

2.10. $Y(f) = \int_{-\infty}^{t} X(v) X(f - v) dv$

$|X(v)| \neq 0$ if $|v| \leq W$

$|X(f - v)| \neq 0$ if $|f - v| \leq W$

$(f - v) \leq W$ for $f \leq W + v$ when $v \geq 0$ and $v \leq W$

$(f - v) \geq -W$ for $f \leq -W + v$ when $v \leq 0$ and $v \geq -W$

$\therefore (f - v) \leq W$ for $0 \leq v \leq W$ when $f \leq 2W$

$(f - v) \geq -W$ for $-W \leq v \leq 0$ when $f \geq -2W$

$\therefore$ Over the range of integration $[-W, W]$, the integral is non-zero if $|f| \leq 2W$

Source - Solutions Manual, for "Communication Systems" 5th Ed, Wiley 2009

2.11 a) Given a rectangular function: $g(t) = \frac{1}{T}\text{rect}\left(\frac{t}{T}\right)$, for which the area under $g(t)$ is always equal to 1, and the height is 1/T.

$$\frac{1}{T}\text{rect}\left(\frac{t}{T}\right) \rightleftarrows \text{sinc}(fT)$$

Taking the limits:

$$\lim_{T\to 0}\frac{1}{T}\text{rect}\left(\frac{t}{T}\right) = \delta(t)$$

$$\lim_{T\to 0}\frac{1}{T}\text{sinc}(fT) = 1$$

b)

$$g(t) = 2W\text{sinc}(2Wt)$$

$$2W\text{sinc}(2Wt) \rightleftarrows \text{rect}\left(\frac{f}{2W}\right)$$

$$\lim_{W\to\infty} 2W\text{sinc}(2Wt) = \delta(t)$$

$$\lim_{W\to\infty}\text{rect}\left(\frac{2}{2W}\right) = 1$$

2.12.

$$G(f) = \frac{1}{2} + \frac{1}{2}\text{sgn}(f)$$

By duality:

$$G(f) \rightleftarrows \frac{1}{2}\delta(-t) - \frac{1}{j2\pi t}$$

$$\therefore g(t) = \frac{1}{2}\delta(t) + \frac{j}{2\pi t}$$

Source - Solutions Manual for "communication systems", 5th Ed, Wiley 2009

~~Since~~

filter

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t).$$

Since filter is stable

$$\int_{-\infty}^{\infty} |h(t)|\,dt < \infty, \text{ and}$$

Since $x(t)$ has finite energy

$$\int_{-\infty}^{\infty} |x(t)|^2\,dt < \infty.$$

$$y(t) = \int h(\tau)\,x(t-\tau)\,d\tau$$ is the filter output.

We need to show that

$$\int_{-\infty}^{\infty} |y(t)|^2\,dt < \infty.$$

By Rayleigh's theorem $\int_{-\infty}^{\infty} |y(t)|^2\,dt = \int_{-\infty}^{\infty} |Y(f)|^2\,df$,

and therefore it suffices to show that

$$\int_{-\infty}^{\infty} |Y(f)|^2\,df < \infty.$$

We have $Y(f) = H(f)\,X(f)$, and

hence $$\int_{-\infty}^{\infty} |Y(f)|^2\,df = \int_{-\infty}^{\infty} |X(f)|^2\,|H(f)|^2\,df.$$

2.15 (continued),

Since

$$H(f) = \int_{-\infty}^{\infty} h(t) e^{-j2\pi f t} dt,$$

we have

$$|H(f)| = \left| \int_{-\infty}^{\infty} h(t) e^{-j2\pi f t} dt \right|$$

$$\leq \int_{-\infty}^{\infty} |h(t)| dt < \infty.$$

Let $\int_{-\infty}^{\infty} |h(t)| dt = c < \infty$,

then using this we have

$|H(f)| \leq c$ for all f, and hence

$$\int_{-\infty}^{\infty} |Y(f)|^2 df = \int_{-\infty}^{\infty} |X(f)|^2 |H(f)|^2 df$$

$$\leq c \int_{-\infty}^{\infty} |X(f)|^2 df$$

$$= c \int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$$

since $x(t)$ has finite energy.

2.16

~~2.17~~.

The transfer function of the summing block is: $H_1(f) = [1 - \exp(-j2\pi fT)]$.

The transfer function of the integrator is: $H_2(f) = \frac{1}{j2\pi f}$

These elements are cascaded :

$$
\begin{aligned}
H(f) &= (H_1(f)H_2(f)) \cdot (H_1(f)H_2(f)) \\
&= -\frac{1}{(2\pi f)^2}[1 - \exp(-j2\pi fT)]^2 \\
&= -\frac{1}{(2\pi f)^2}[1 - 2\exp(-j2\pi fT) + \exp(-j4\pi fT)]
\end{aligned}
$$

Source: Solutions Manual to "Communication Systems", 5th Ed, Wiley 2009

2.18. $y(t) = \int_{t-T}^{t} x(\tau)\,d\tau$ —— (1)

a) let $\tau' \triangleq t-\tau$, then we have

$$\int_{t-T}^{t} x(\tau)\,d\tau = \int_{0}^{T} x(t-\tau')\,d\tau'$$

$$= \int_{0}^{T} h(\tau')\,x(t-\tau')\,d\tau'$$

where $h(\tau') = \begin{cases} 1 & 0 \le \tau' \le T \\ 0 & \text{otherwise} \end{cases}$ —— (2)

From (2) we see that the output $y(t)$ is obtained by filtering $x(t)$ with a filter having impulse response $h(t) = \begin{cases} 1, & 0 \le t \le T \\ 0, & \text{otherwise.} \end{cases}$ —— (3)

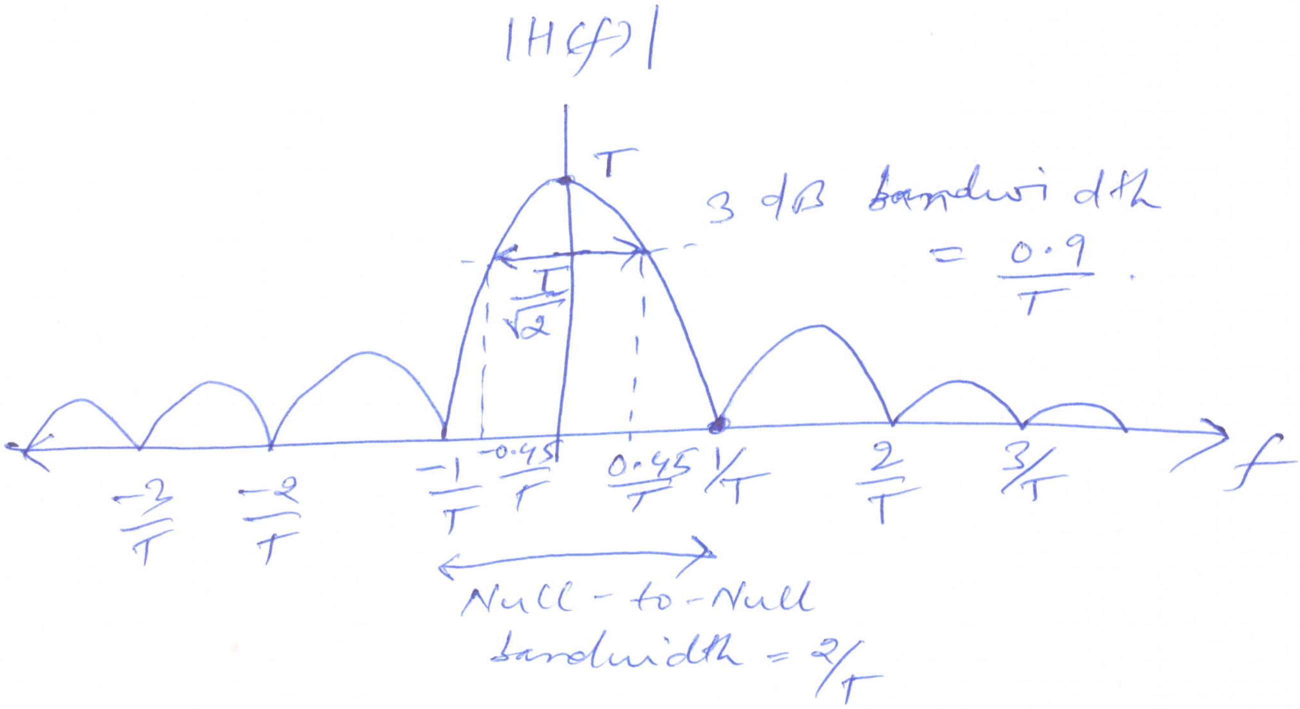
∴ The ~~trans~~ transfer function of this filter is

$$H(f) = \int_{-\infty}^{\infty} h(t)\,e^{-j2\pi f t}\,dt$$

$$= \int_{0}^{T} e^{-j2\pi f t}\,dt$$

$$= T\,\mathrm{sinc}(fT)\,e^{-j\pi f T}$$

b). Note that the filter in part a) is a low pass filter



The filter in a) cannot exist in reality since it is physically impossible to realize a ~~de~~ filter whose impulse response has discontinuity at $t=0$ and $t=T$.

Therefore in part b) we try to approximate the ideal low pass filter $h()$ in part a), with a RC filter as shown in the next page.

2.18 b) Continued.

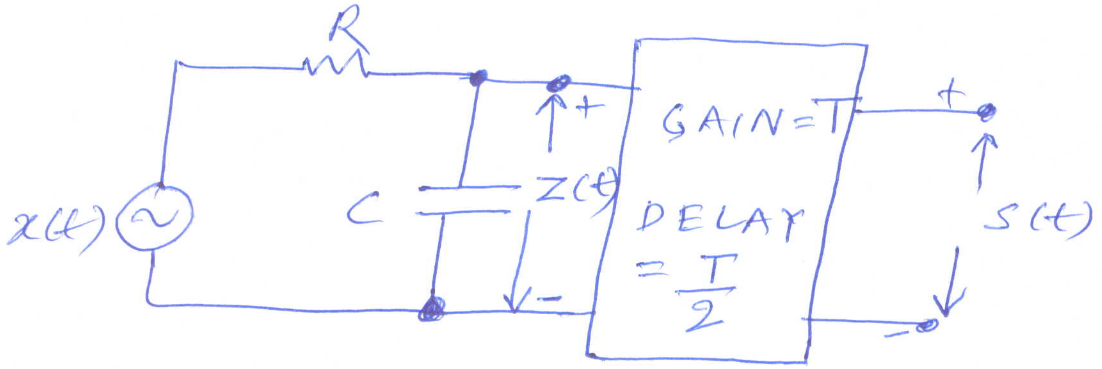


Fig: An RC low pass filter (LPF) approximation to the LPF in part (a).

From the figure above it is clear that

$$V(f) \triangleq \frac{S(f)}{X(f)} = \frac{T}{RC}\left[\frac{1}{\frac{1}{RC}+j2\pi f}\right]e^{-j\pi f T} \quad \text{—} \ (4)$$

We would like to find

$$v(t) = \int V(f)\, e^{j2\pi f t}\, df$$

(i.e., the impulse response of the LTI system shown above).

We know that

$$e^{-kt}u(t) \xrightleftharpoons{\text{Fourier}} \frac{1}{(k+j2\pi f)} \quad \text{—} \ (5)$$

where $u(t) = \begin{cases} 1, & t > 0 \\ 0, & \text{otherwise} \end{cases}$

is the unit step function.

Comparing (4) and (5) we get.

$$V(t) = \frac{T}{RC} e^{-\frac{(t - T/2)}{RC}} u\left(t - \frac{T}{2}\right) \quad \text{---} (6)$$

If $x(t) = u(t)$ (i.e., we apply a step input), then the output of the filter in part (a) would be

$$y(t) = \int_{t-T}^{t} x(\tau)\, d\tau = \int_{t-T}^{t} u(\tau)\, d\tau .$$

The output at time $t=T$ would therefore be

$$y(t=T) = \int_{0}^{T} u(\tau)\, d\tau = T \quad \text{---} (7)$$

The output of the approximate filter in part (b) would be

$$s(t) = \int_{-\infty}^{\infty} V(\tau)\, u(t-\tau)\, d\tau$$

$$\therefore s(t=T) = \int_{-\infty}^{\infty} V(\tau)\, u(T-\tau)\, d\tau$$

$$= \frac{T}{RC} \int_{-\infty}^{\infty} e^{-\frac{(t-T/2)}{RC}} u(t-T/2)\, u(T-\tau)\, d\tau$$

(using (6))

$$= \frac{T}{RC} \int_{T/2}^{T} e^{-\frac{(t-T/2)}{RC}} d\tau \; = \underline{T\left(1 - e^{-T/2RC}\right)}$$

(8)

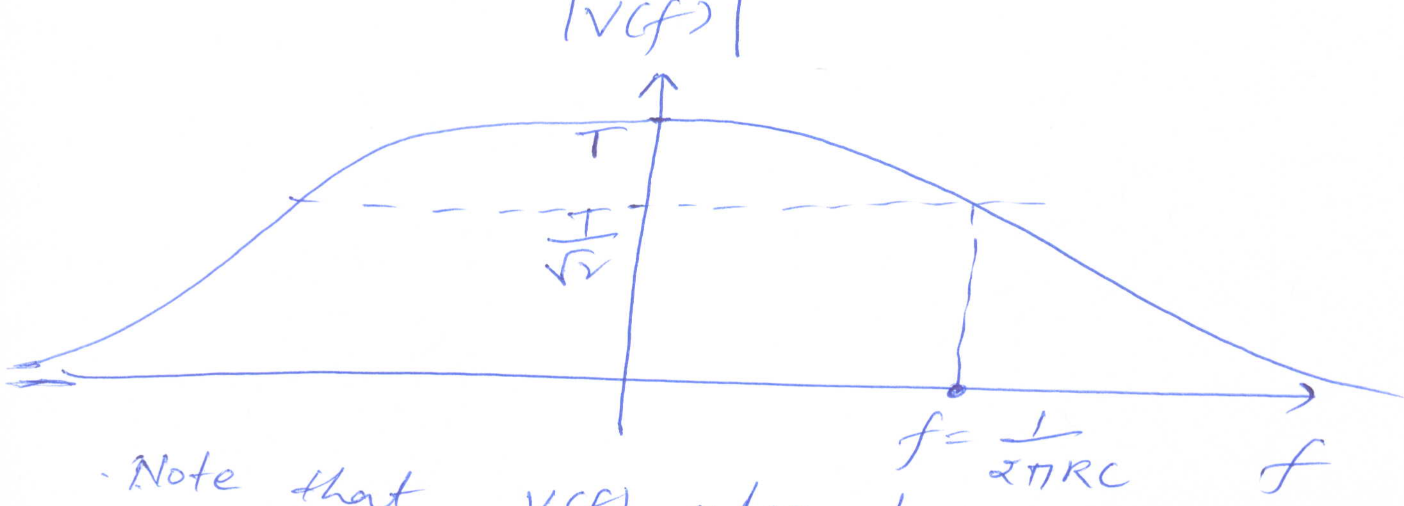
Comparing (7) and (8) we see that

$$\left|\frac{s(t=T) - y(t=T)}{y(t=T)}\right| = e^{-T/2RC}$$ (absolute (relative error))

This error will be small, if R, C satisfy $RC \ll T$.

From (5) we have

$$|V(f)| = \frac{T}{\sqrt{1+(2\pi f RC)^2}}$$



Note that V(f) also has a low pass characteristic, and its 3-dB bandwidth $= \frac{1}{\pi RC}$.

The RC-filter approximation of the filter in part (a) is therefore expected to be good if the 3-dB bandwidths of both the low pass filters is of the same order, i.e,

$$\frac{1}{T} \approx \frac{1}{\pi RC}.$$

2.20

$H(f)$ is given by

$$|H(f)| = \begin{cases} 1 & , |f-f_c| < B \\ 1 & , |f+f_c| < B \\ 0 & , \text{otherwise}, \end{cases}$$

$$\angle H(f) = \begin{cases} -2\pi(f-f_c)t_0 & , |f-f_c| < B \\ -2\pi(f+f_c)t_0 & , |f+f_c| < B \end{cases}$$

$$H(f) = |H(f)|\, e^{j\angle H(f)} \qquad \text{1}$$

$$x(t)u(t) \longrightarrow \boxed{\begin{matrix}\text{FILTER} \\ h(t)\end{matrix}} \longrightarrow y(t)$$

let $z(t) \triangleq x(t)u(t) = \begin{cases} A\cos 2\pi f_0 t & , t > 0 \\ 0 & , t \leq 0 \end{cases}$

$$X(f) = \text{Fourier}\,(A\cos 2\pi f_0 t)$$
$$= \frac{A}{2}\left[\delta(f-f_0) + \delta(f+f_0)\right],$$

and

$$U(f) = \text{Fourier}\,(u(t))$$
$$= \frac{1}{j2\pi f} + \frac{\delta(f)}{2}. \qquad \text{2}$$

2.20 (continued) . . .

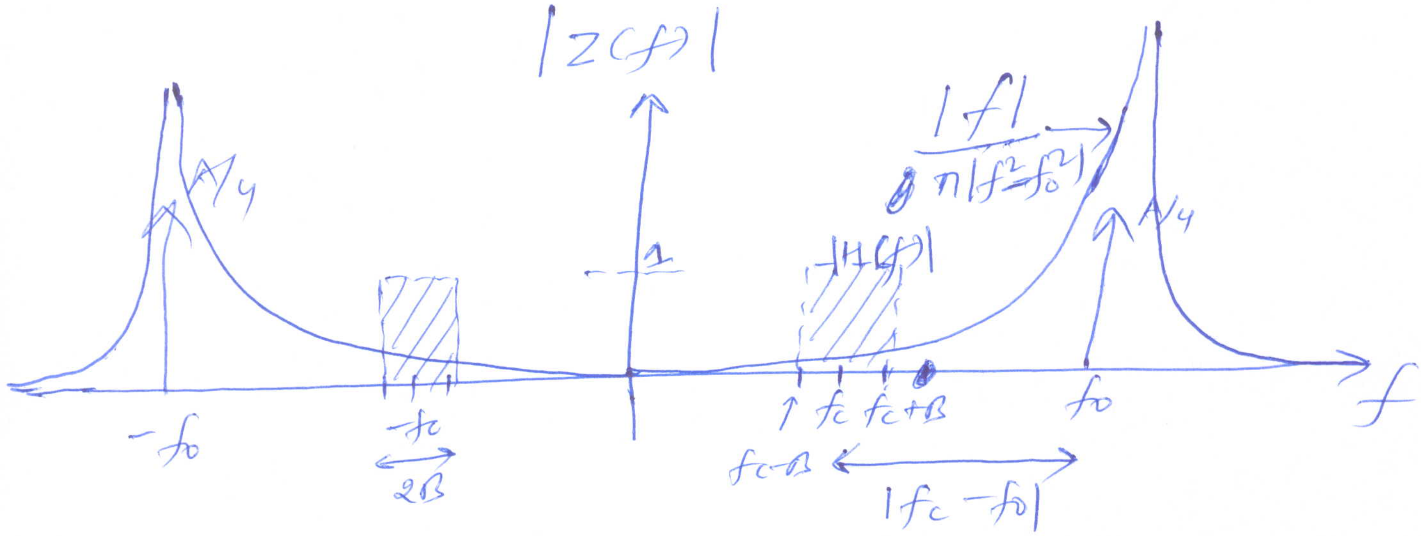
$$Z(f) = \text{Fourier}\left(x(t)\, u(t) \right)$$

$$= X(f) \circledast U(f)$$

↑ convolution

$$= \frac{A}{2}\left[\frac{\delta(f-f_0)}{2} + \frac{\delta(f+f_0)}{2} + \frac{1}{j2\pi(f-f_0)} + \frac{1}{j2\pi(f+f_0)} \right]$$

$$= \frac{A}{2}\left[\frac{f}{j\pi(f^2-f_0^2)} + \frac{\delta(f-f_0)}{2} + \frac{\delta(f+f_0)}{2} \right]$$



Since $|f_c - f_0| >> B$, from the figure we can see that in the passband of the filter (almost constant amplitude)

$$Z(f) \approx \begin{cases} \frac{A}{2}\frac{f_c}{j\pi(f_c^2-f_0^2)}, & |f-f_c|<B \\ \frac{-A}{2}\frac{f_c}{j\pi(f_c^2-f_0^2)}, & |f+f_c|>B \end{cases}$$

—— 3

2.20 (continued) using ① and ③

$\therefore$ $Y(f) = H(f)\, Z(f)$ is approximately given by

$$Y(f) \approx \begin{cases} \dfrac{A f_c\, e^{-j2\pi(f-f_c)t_0}}{j2\pi(f_c^2-f_0^2)}, & |f-f_c|<B \\[2ex] \dfrac{-A f_c\, e^{-j2\pi(f-f_c)t_0}}{j2\pi(f_c^2-f_0^2)}, & |f+f_c|<B. \end{cases}$$

$$\therefore\ y(t) = \int_{-\infty}^{\infty} Y(f)\, e^{j2\pi f t}\, df$$

$$= \frac{A f_c}{j2\pi(f_c^2-f_0^2)} \int_{f_c-B}^{f_c+B} e^{-j2\pi(f-f_c)t_0}\, df$$

$$\frac{-A f_c}{j2\pi(f_c^2-f_0^2)} \int_{-f_c-B}^{-f_c+B} e^{-j2\pi(f-f_c)t_0}\, df$$

$\therefore$

$$y(t) = \frac{2 A f_c B}{\pi(f_c^2-f_0^2)} \operatorname{sinc}(2B(t-t_0)) \sin 2\pi f_c t$$

Note that $y(t) \to 0$, as $t \to \infty$.

Also due to the term $\operatorname{sinc}(2B(t-t_0))$ it is clear that $y(t)$ goes to zero at a faster rate if B is large.