Classical waves

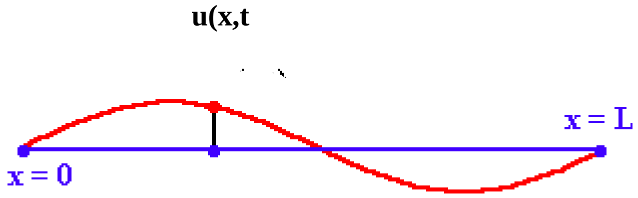


Figure 2.1: A vibrating string

A linear partial differential equation

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u(x,t)}{\partial t^2}$$

Boundary conditions

$$u(0,t) = 0 \qquad \text{and} \qquad u(l,t) = 0 \qquad \text{at all times}$$

Separation of variables: A technique used when the two variables are independent

$$u(x,t) = X(x)T(t)$$

which gives

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = \frac{1}{v^2T(t)}\frac{d^2T(t)}{dt^2}$$

Since LHS is only dependent on the position and RHS on time they must be equal to a constant, K

$$\frac{d^2X(x)}{dx^2} - KX(x) = 0 \qquad and \qquad \frac{d^2T(t)}{dt^2} - Kv^2T(t) = 0$$

These are linear differential equations with constant coefficients.

Solutions $K = 0, K < 0, K > 0$

For $K = 0$, the solution are trivial – no use

For $K > 0$, its a subset of the solution for $K < 0$

For $K < 0$

Lets rewrite the equation in this form $\frac{d^2y}{dx^2} + k^2y = 0$ where $K = -\beta^2$

We need a solution that when differentiated twice gives back the same function. Lets try $y = e^{\alpha x}$

This gives, $(\alpha^2 + \beta^2)\, y(x) = 0$

i.e. $\alpha = \pm\, i\,\beta$

The general solution is then

$$y(x) = c_1 e^{i\beta x} + c_2 e^{-i\beta x}$$

Using Euler's formula,

$$e^{\pm i\theta} = cos\theta \pm i \sin\theta$$

we get

$$y(x) = A\cos\beta x + B\sin\beta x$$

See for yourself that the case of $K > 0$ is a special case of the solutions for $K < 0$.

Boundary conditions: See the two equations for $X(x) and\ T(t)$

$X(0) = 0$ implies $A = 0$

$X(l) = 0$ implies $X(l) = B\sin\beta l = 0$ $B = 0$ is trivial. So, $\sin\beta l = 0$ gives

$$\beta l = n\pi \qquad n = 1,2,3 \ldots$$

$n = 0$ is not a solution because the wave does not exist.

$$X(x) = B\sin\frac{n\pi x}{l}$$

Also, for $T(t)$

$$\frac{d^2T(t)}{dt^2} + \beta^2 v^2 T(t) = 0$$

The general solution (remember $\beta = n\pi/l$) is

$$T(t) = D\cos\omega_n t + E\sin\omega_n t$$

So the amplitude of the wave u is given by (it now depends on n)

$$u_n(x,t) = X(x)T(t) = \left(B_n \sin\frac{n\pi x}{l}\right)(D_n\cos\omega_n t + E_n\sin\omega_n t)$$
$$= (F_n\cos\omega_n t + G_n\sin\omega_n t)\sin\frac{n\pi x}{l}$$

Superposition

As each $u_n(x,t)$ is a solution to the linear differential equation, so is any sum of the $u_n(x,t)'s$.

A most general solution is

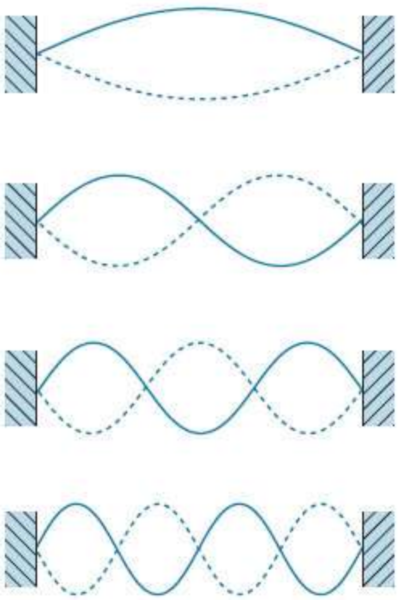
$$u(x,t) = \sum_{n=1}^{\infty}(F_n\cos\omega_n t + G_n\sin\omega_n t)\sin\frac{n\pi x}{l}$$

or

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos(\omega_n t + \phi_n) \sin\frac{n\pi x}{l}$$

where A is the amplitude and ϕ the phase angle. No matter how the string is plucked its shape will evolve according to the above equations.

Each $u_n(x,t)$ is called a normal mode. The time dependence of each normal mode represents harmonic motion of frequency $\nu_n = \omega_n/2\pi \; = vn/2l$ (since $\omega_n = n\pi v/l$)



mode	wavelength	frequency
first	$2L$	$\frac{v}{2L}$
second	L	$\frac{v}{L}$
third	$\frac{2L}{3}$	$\frac{3v}{2L}$
fourth	$\frac{L}{2}$	$\frac{2v}{L}$

Figure 1.2: The first few modes of vibration

First harmonic = fundamental mode, frequency $v/2l$

Second harmonic = first overtone, frequency v/l

Mid-point of the second harmonic does not change with time. Its fixed at zero. This is a **node** which you will also encounter in quantum mechanics. $x = 0$ and $x = l$ are not nodes – they are boundary conditions.

These are standing waves.

Add up the first two harmonics, phase shifted by 90°

$$u(x,t) = \cos\omega_1 t \sin\frac{\pi x}{l} + \frac{1}{2}\cos\left(\omega_2 t + \frac{\pi}{2}\right)\sin\frac{2\pi x}{l}$$

Some work: sketch the travelling wave.

Schrödinger Equation

Lets start with the classical wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$$

which as we have seen above can be solved to give

$$u(x,t) = \psi(x)\cos\omega t$$

$\psi(x)$ is called the spatial amplitude of the wave. This gives,

$$\frac{d^2\psi}{dx^2} + \frac{\omega^2}{v^2}\,\psi(x) = 0$$

Since $\omega = 2\,\pi\nu$ and $\nu\lambda = v$,

$$\frac{d^2\psi}{dx^2} + \frac{4\pi^2}{\lambda^2}\,\psi(x) = 0$$

Now the total energy

$$E = KE + PE = \frac{p^2}{2m} + V(x)$$

and so

$$p = \{2m[E - V(x)]\}^{1/2}$$

Use de Broglie relation $\lambda = h/p$. This is where quantum mechanics comes in

$$\lambda = \frac{h}{p} = \frac{h}{\{2m[E - V(x)]\}^{\frac{1}{2}}}$$

and we get,

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}\,[\,E - V(x)]\,\psi(x) = 0$$

or

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi(x) = E\,\psi(x)$$

This is the time-independent Schrödinger equation

In this course we will not worry about the time-dependent Schrödinger equation

Operators

An operator operates – it does something. For example, an "turn 90° left" is an operator that tells us to turn left by 90°. Another example could be "walk five paces ahead".

Mathematical operators tell us to perform a mathematical operation on a function $(f(x)$ to give another function $g(x)$.

Some examples of mathematical operators

INTEGRATE: $\int_0^1 f(x) = g(x)$

SQR: $\left(f(x)\right)^2 = g(x)$

DIFFERENTIATE: $\frac{d}{dx} f(x) = g(x)$

In general we can denote an operator using a hat on it,

$$\hat{A}\, f(x) = g(x)$$

Operators and quantum mechanics

In quantum mechanics, we encounter only linear operators. This is one of the postulates of QM which we will discuss later.

$$\hat{A}\,[c_1 f_1(x) + c_2\, f_2(x)] = c_1\, \hat{A}(f_1(x) + c_2 \hat{A}(\, f_2(x)$$

Here c_1 and c_2 can be complex numbers.

Differentiate and integrate are linear. Squaring is non-linear.

Operators may not commute like numbers, i.e. $\hat{A}\,\hat{B}\, f(x)$ is not necessarily equal to $\hat{B}\hat{A}\, f(x)$. As an example consider the case of a person walking five paces and turning 90°.

Eigenfunctions and Eigenvalues

A function that gets operated and results in the same function apart from a multiplicative factor is an eigenfunction of the operator

$$\hat{A}\, f(x) = a\, f(x)$$

Finding the eigenfunction of the operator and the eigenvalue is called an eigenvalue problem.

The Schrödinger equation can be written as

$$\left[-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\right]\psi(x) = E\,\psi(x)$$

or

$$\hat{H}\,\psi(x) = E\,\psi(x)$$

where $\hat{H}$ is called the Hamiltonian operator and the eigenvalue is the energy. So there is a correspondence between the operator and a measurable. Such correspondences between operators and classical-mechanical variables are fundamental to the formalism of QM.

Since the energy is KE + PE, and $\widehat{PE} = V(x) \therefore \widehat{KE} = -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}$

$$KE = \frac{p^2}{2m} \Rightarrow \hat{p}^2 = -\hbar^2\frac{d^2\psi}{dx^2}$$

$$\therefore \hat{p}.\hat{p} = -\hbar^2\frac{d^2\psi}{dx^2} \; or \; \hat{p} = -i\hbar\frac{d}{dx} \quad \text{(the minus sign is needed for the correct direction)}$$

Probability

Discrete Events

An experiment has n possible outcomes, each with probability p_j. We perform the experiment a large number of times (ideally infinite number of times)

$$p_j = \lim_{n\to\infty}\frac{N_j}{N} \qquad\qquad j = 1, 2, 3 \ldots\ldots n$$

$$0 \le p_j \le 1 \qquad \text{and} \qquad \Sigma p_j = 1 \quad \text{(normalization)}$$

Suppose we get a value x_j at the j^{th} experiment, then the average is defined as

$$\langle x\rangle = \Sigma\, x_j p_j = \Sigma\, x_j p(x_j)$$

Second momemt

$$\langle x^2\rangle = \sum_{j=1}^{n} x_j^2\, p_j$$

Second central moment or variance

$$\sigma_x^2 = \langle (x - \langle x\rangle)^2\rangle = \sum_{j=1}^{n}\left(x_j - \langle x\rangle\right)^2 p_j = \langle x^2\rangle - \langle x\rangle^2$$

σ_x is called the standard deviation.

Continuous distributions

$$prob\ (x, x + dx) = p(x)dx$$

$$prob\ (a \le x \le b) = \int_a^b p(x)d(x)$$

Normalization condition

$$\int_{-\infty}^{\infty} p(x)d(x) = 1$$

Average and standard deviation

$$\langle x \rangle = \int_{-\infty}^{\infty} x\, p(x)d(x)$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2\, p(x)d(x)$$

$$\sigma_x^2 = \int_{-\infty}^{\infty} (x - \langle x \rangle)^2\, p(x)d(x)$$

Quantum mechanics and probability

If we restrict the particle in a certain region, then the probability of finding the particle in this region is one. Outside this region the particle does not exist. Since the intensity of a wave is the square of the magnitude of the amplitude, mathematically we say $\psi^*(x)\psi(x)dx$ is the probability that the particle is located between x and $x + dx$.