Particle on a ring - 2D rotation

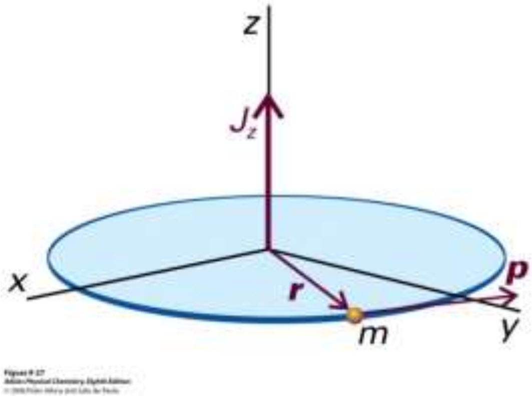


Figure 6.1: The angular momentum

Moment of inertia: $I = mr^2$

Energy, $E = J_z^2/2I$

$J_z = \pm pr$ and $p = h/\lambda$ (de Broglie). This gives $J_z = \pm hr/\lambda$

- Two directions of rotation
- shorter the wavelength, greater the angular momentum

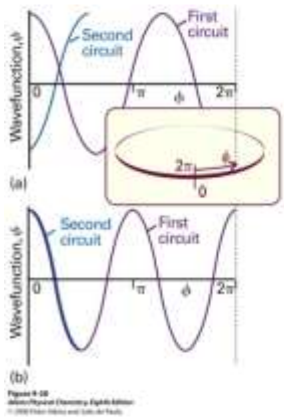


Figure 6.2: Constructive and destructive interference

The above figure gives, $\lambda = 2\pi r/m_l$

Therefore, $J_z = m_l\hbar \qquad m_l = 0, \pm 1, \pm 2, \ldots$

Energy, $E = m_l^2\hbar^2/2I$

Quantum mechanically

We solve the Schrödinger equation. Particle in a plane. The potential is zero.

$$\hat{H} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

Symmetry of the system suggests cylindrical coordinates

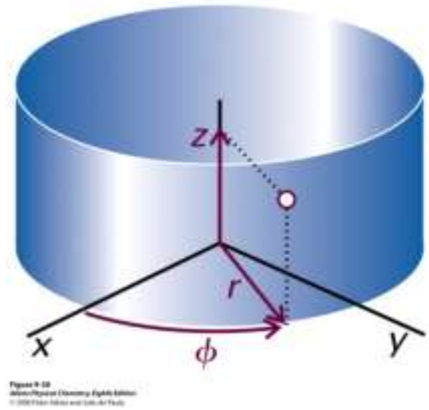


Figure 6.3: Cylindrical coordinate system. $x = r\cos\phi$, $y = r\sin\phi$, $z = z$. In the present case we do not use the z-coordinate. $r = \sqrt{x^2 + y^2}, \phi = \tan^{-1} \mathrm{y/x}$

In this system, the Hamiltonian is

$$\hat{H} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}\right)$$

(One needs to use $\frac{\partial f}{\partial x} = \frac{\partial r}{\partial x}.\frac{\partial f}{\partial r} + \frac{\partial \phi}{\partial x}.\frac{\partial f}{\partial \phi}$ for the derivation, which we will not worry about)

$$\hat{H} = -\frac{\hbar^2}{2m}\left(\frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}\right) = -\frac{\hbar^2}{2I}\left(\frac{\partial^2}{\partial \phi^2}\right)$$

S.E. is

$$\frac{\partial^2 \psi}{\partial \phi^2} = -\frac{2IE}{\hbar^2}\,\psi$$

whose solutions are

$$\psi_{m_l}(\phi) = \frac{e^{i m_l \phi}}{\sqrt{2\pi}} \qquad \text{where} \qquad m_l = \pm\frac{\sqrt{2IE}}{\hbar}$$

(Till now m_l is not an integer. It is just a constant)

Boundary conditions: cyclic $\psi(\phi + 2\pi) = \psi(\phi)$

$$\psi_{m_l}(\phi + 2\pi) = \frac{e^{im_l(\phi+2\pi)}}{\sqrt{2\pi}} = \frac{e^{im_l\phi}.e^{2\pi i m_l}}{\sqrt{2\pi}} = \psi_{m_l}(\phi)e^{2\pi i m_l} = (-1)^{2m_l}\,\psi(\phi)$$

($e^{i\pi} = -1$). We require an equality in the above equation. This gives, $(-1)^{2m_l} = 1$. Or in other words, $2m_l$ must be an even integer (+ve or -ve). Or m_l must be an integer. $m_l = 0, \pm 1, \pm 2, \ldots$

Wavefunctions

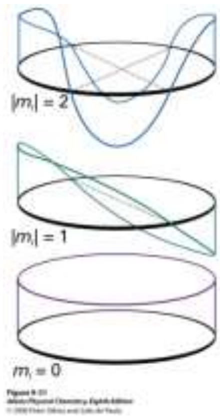


Figure 6.4: Real part of the wavefunctions - particle on a ring

The wavelength of the wavefunction with $m_l = 0, \psi = 1/\sqrt{2\pi}$ is infinite.

$\boxed{E_{m_l} = m_l^2\hbar^2/2I}$ implies energy does not depend on the direction of rotation

$\boxed{J_z = m_l\hbar}$ implies $\hbar$ is the fundamental unit of angular momentum

Quantization of rotation

$$\boldsymbol{l} = \boldsymbol{r} \times \boldsymbol{p} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$$

The z-component of angular momentum, $l_z = xp_y - yp_x$ is represented by the operator

$$\widehat{l_z} = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

In terms of r, ϕ

$$l_z = \frac{\hbar}{i}\frac{\partial}{\partial \phi}$$

Now operate with the angular momentum operator to find the eigenvalues.

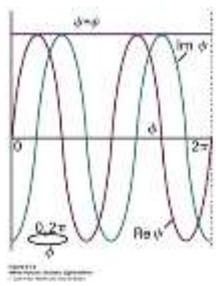
Uncertainty in the position of the particle

We have defined the angular momentum completely so the angle (which corresponds to the position) remains undefined. The particle could be anywhere on the ring.

The probability density of finding the particle $\psi_{m_l}^* \psi_{m_l} = 1/2\pi$ which is independent of the angle ϕ.

Angular momentum - Angle: complementary observables

Momentum-Position: complementary observables



Spherical polar coordinates

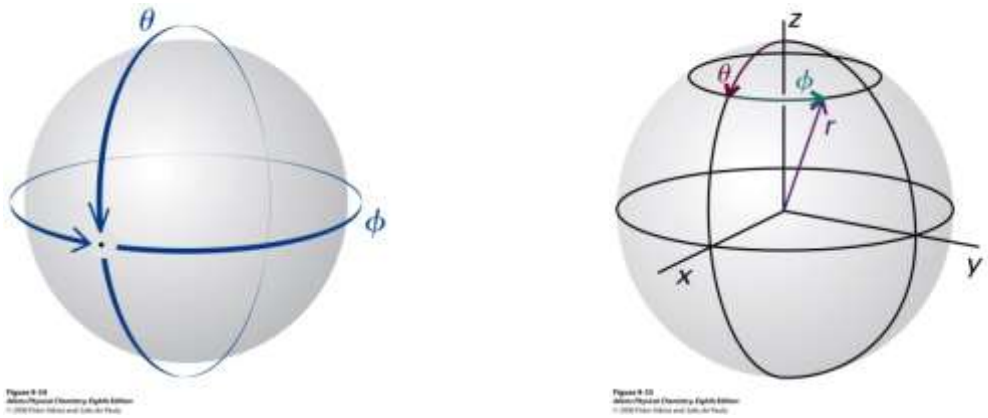


Figure 6.5: Spherical coordinates

Particle moves on the surface of the sphere

Hamiltonian $\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V$ $\quad\quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

ψ is a function of *colatitute*, θ and *azimuth*, ϕ.

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\theta,\phi) = E\psi(\theta,\phi)$$

Laplacian in spherical polar coordinates

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Lambda^2$$

where, the *legendrian*, Λ^2 is given by

$$\Lambda^2 = \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}$$

Since ψ does not depend on r

$$\frac{1}{r^2}\Lambda^2\psi = -\frac{2mE}{\hbar^2}\psi$$

$$\Lambda^2\psi = -\epsilon\psi \qquad \epsilon = \frac{2IE}{\hbar^2}$$

Separation of variables [use $\psi(\theta,\phi) = \Theta(\theta).\Phi(\phi)$]

$$\frac{1}{\phi}\frac{d^2\Phi}{d\phi^2} = -m_l^2 \qquad \text{and} \qquad \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\sin\theta\frac{d\Theta}{d\theta} + \epsilon\sin^2\theta = m_l^2$$

The first one we have seen already

The second gives solutions known as a*ssociated Legendre functions*. The cyclic boundary counditions on Θ give rise to another quantum number, l. Since both m_l and l are present in the second equation, it implies they should be related. The values of m_l (the ***magnetic quantum number***) are governed by the value of l (the ***orbital angular momentum quantum number***)

$l = 0, 1, 2, \ldots \qquad m_l = l, l-1, \ldots\ldots, -l \qquad (2l+1)$ values

$Y_{l,m}(\theta,\phi)$ are called the spherical harmonics

The solution gives the energy as

$$E = l(l+1)\frac{\hbar^2}{2I} \qquad l = 0, 1, 2, \ldots$$

Table 9.3 The spherical harmonics

l	m_l	$Y_{l,m_l}(\theta,\varphi)$
0	0	$\left(\frac{1}{4\pi}\right)^{1/2}$
1	0	$\left(\frac{3}{4\pi}\right)^{1/2}\cos\theta$
	±1	$\mp\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta\, e^{\pm i\phi}$
2	0	$\left(\frac{5}{16\pi}\right)^{1/2}(3\cos^2\theta-1)$
	±1	$\mp\left(\frac{15}{8\pi}\right)^{1/2}\cos\theta\sin\theta\, e^{\pm i\phi}$
	±2	$\left(\frac{15}{32\pi}\right)^{1/2}\sin^2\theta\, e^{\pm 2i\phi}$
3	0	$\left(\frac{7}{16\pi}\right)^{1/2}(5\cos^3\theta-3\cos\theta)$
	±1	$\mp\left(\frac{21}{64\pi}\right)^{1/2}(5\cos^2\theta-1)\sin\theta\, e^{\pm i\phi}$
	±2	$\left(\frac{105}{32\pi}\right)^{1/2}\sin^2\theta\cos\theta\, e^{\pm 2i\phi}$
	±3	$\mp\left(\frac{35}{64\pi}\right)^{1/2}\sin^3\theta\, e^{\pm 3i\phi}$

Table 9-3
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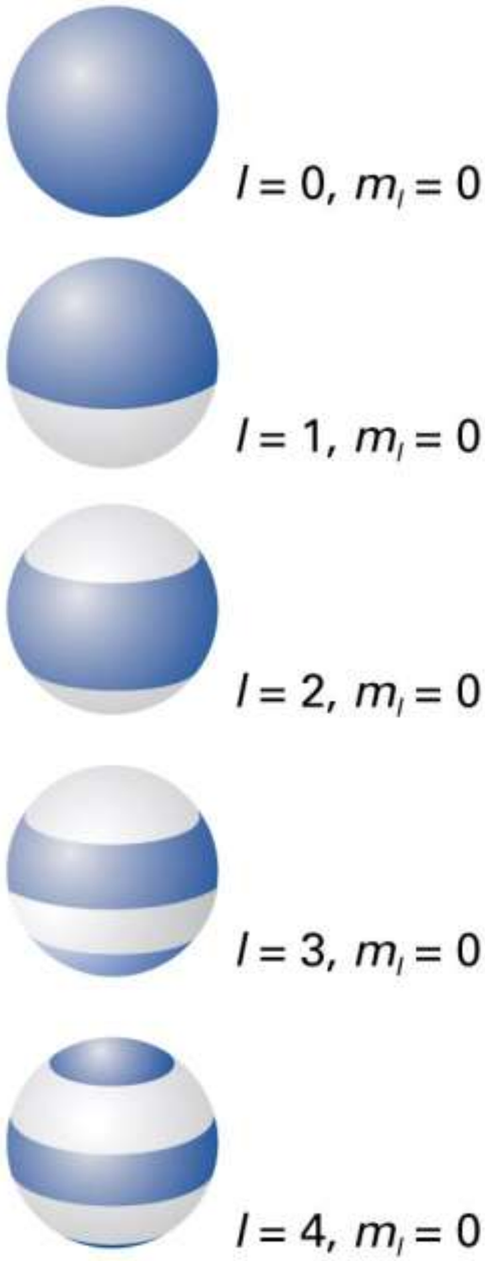

Figure 9-36
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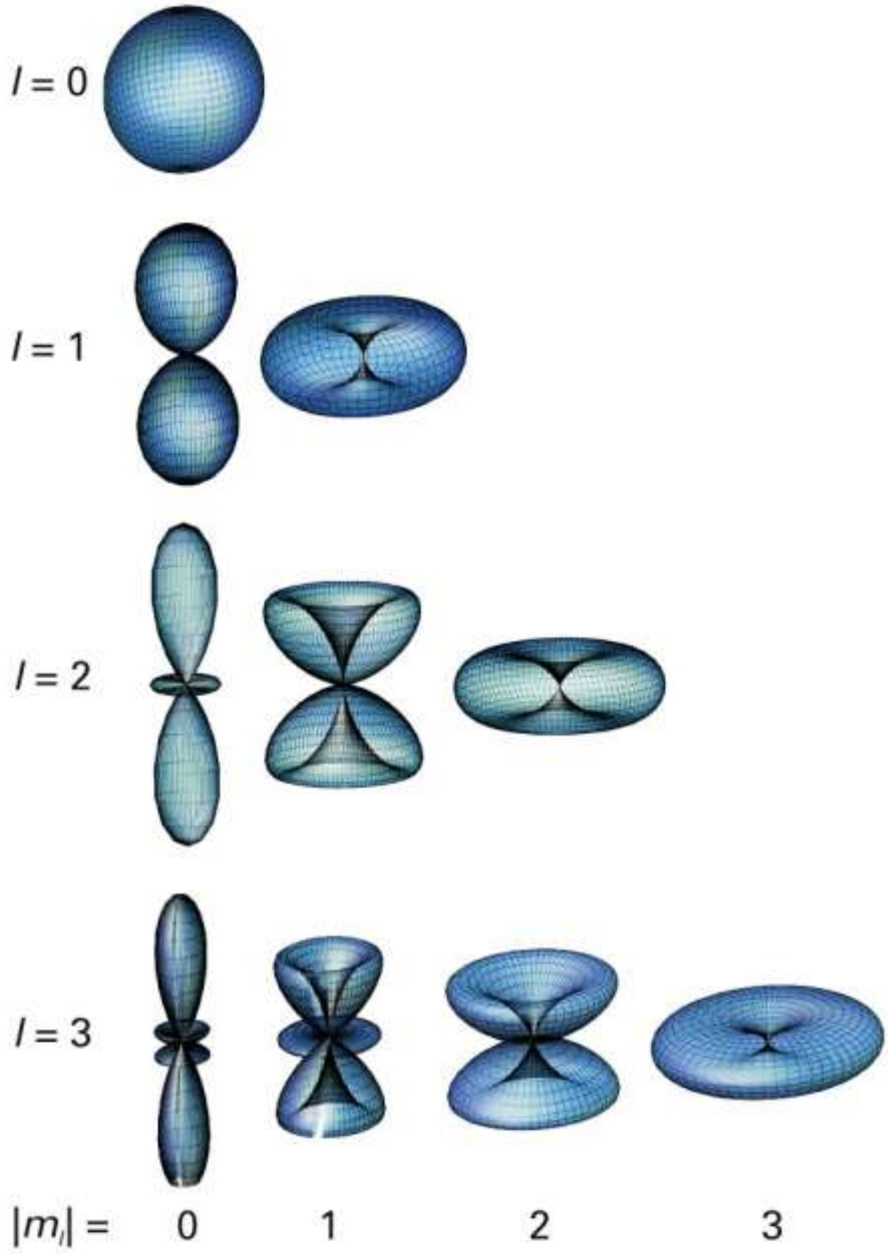

Figure 9-37
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