Name: ——————————————————Group No:——— Entry No: ——————

DEPARTMENT OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY DELHI
MAJOR TEST 2018-2019 SECOND SEMESTER
MTL101 (LINEAR ALGEBRA AND DIFFERENTIAL EQUATIONS)

Time: 2 hours Max. Marks: 40

Write your Name and Group number and Entry number at the places specified above. Attempt all questions. All notations are standard. All parts of a question must be answered at one place. Exhibit clearly all the steps. Use of any electronic gadget including calculator is NOT allowed. Attach question paper with the answer book. Do not do any rough work on the question paper. No query will be entertained.

1a. Suppose that $\{x_1, \ldots\ldots\ldots, x_n\}$ is a linearly independent set of vectors in a vector space. Let $x = \sum_{i=1}^{n} \beta_i x_i$, $\beta_i \neq 0\ \forall i, 1 \leq i \leq n$. Find the condition under which $\{x_1 - x, \ldots\ldots\ldots, x_n - x\}$ is linearly dependent. (3)

1b. Let $V = \{p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \quad : \quad a_0, a_1, a_2, a_3 \in R\}$ be a vector space. Let $T : V(R) \longrightarrow V(R)$ be a linear transformation defined by

$$T(p(x)) = a_0 + a_1(1+x) + a_2(1+x)^2 + a_3(1+x)^3.$$

What is the matrix of T with respect to ordered basis $B = \{1, (1+x), (1+x^2), (1+x^3)\}$? (4)

2a. Let a linear transformation $T : R^{4\times 1} \to R^{4\times 1}$ be defined by $TX = AX$ where

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}.$$

Find a basis for the null space of T. (3)

2b. Find the values of λ and k for which the following system

$$\begin{aligned} x + y + z &= 6 \\ x + 2y + 3z &= 10 \\ x + 2y + \lambda z &= k \end{aligned}$$

has (i). no solution (ii). unique solution (iii). infinite number of solutions (4)

3. Find general solution of the system of differential equations $Y' = AY + b(x)$ where

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}, \quad b(x) = \begin{bmatrix} 2e^{-x} \\ 3x \end{bmatrix}$$

by determining the eigenvalues and eigenvectors of A. (5)

P.T.O.

4a. Use Frobenius method of extended power series to find complete solution of $xy'' + y' + y = 0$. (5)

4b. Show that between any two positive zeros of $J_{n+1}(x)$ there is precisely one zero of $J_n(x)$. (3)

5a. Find the eigenvalues and eigenfunctions of the Sturm-Liouville problem

$$((x^2+1)y')' + \lambda(x^2+1)^{-1}y = 0 \quad y(0) = y(1) = 0,$$

where $\lambda > 0$. (4)

5b. Examine whether the set of functions $\{\cos\frac{n\pi x}{L}\}$, $\quad L > 0, \quad n = 0, 1, 2, \ldots\ldots$ is orthogonal or not with respect to the weight function $r(x) = 1$ on the interval $[-L, L]$. If so, find the orthonormal set. (3)

6a. Let $f(t)$ be a function that is piecewise continuous on every finite interval in the range $t \geq 0$ and satisfies $\mid f(t) \mid \leq Me^{kt}$ for all $t \geq 0$ and for some constants k and M. Then prove or disprove that the Laplace transform of $f(t)$ exists for all $s > k$ where s is the Laplace transform parameter. (3)

6b. Find Laplace transform of the function $\frac{e^{-t}}{\sqrt{t}}$. Hence Show that $L^{-1}(\frac{1}{s\sqrt{s+1}}) = erf(\sqrt{t})$ where $erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$. (3)

MTL 101 Major Brief solutions

1(a). $\{x_1 - x, \ldots, x_n - x\}$ is L.D. if there exist scalars $\alpha_1, \ldots, \alpha_n$ not all zero such that $\sum_{i=1}^{n} \alpha_i (x_i - x) = 0$

$$\text{i.e } \sum_{i=1}^{n} \alpha_i x_i - \left(\sum_{i=1}^{n} \alpha_i\right) x = 0 \longrightarrow (1)$$

Claim: $\sum_{i=1}^{n} \alpha_i \neq 0$

If $\sum_{i=1}^{n} \alpha_i = 0$, then from (1), $\alpha_i = 0, \forall i$, since $x_1, \ldots, x_n$ are L.I.

So we discard this case.

If $\sum_{i=1}^{n} \alpha_i \neq 0$, then from (1), $\quad x = \dfrac{\sum \alpha_i x_i}{\sum \alpha_i}$

$$\Rightarrow \frac{\alpha_i}{\sum_{i=1}^{n} \alpha_i} = \beta_i$$

$$\Rightarrow \sum_{i=1}^{n} \beta_i = 1.$$

1(b).

$$T\left(\sum_{i=0}^{3} a_i x^i\right) = \sum_{i=0}^{3} a_i (x+1)^i$$

Let $B = \{V_1 = 1,\ V_2 = 1+x,\ V_3 = 1+x^2,\ V_4 = 1+x^3\}$

$$T(V_1) = 1\cdot V_1 + 0\cdot V_2 + 0\cdot V_3 + 0\cdot V_4$$
$$T(V_2) = 1\cdot V_1 + 1\cdot V_2 + 0\cdot V_3 + 0\cdot V_4$$
$$T(V_3) = -1\cdot V_1 + 2\cdot V_2 + 1\cdot V_3 + 0\cdot V_4$$
$$T(V_4) = -5\cdot V_1 + 3\cdot V_2 + 3\cdot V_3 + 1\cdot V_4$$

$$[T]_B = \begin{bmatrix} 1 & 1 & -1 & -5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

2(a).

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$Tx = 0 \Rightarrow Ax = 0 \Rightarrow \begin{matrix} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 3x_4 = 0 \end{matrix} \quad \therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_3 + 2x_4 \\ -2x_3 - 3x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

$\Rightarrow \left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}$ spans Null space of T

Since $\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}$ is L.I.

Therefore this set forms a basis of Null space of T.

2(b) $$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & k \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & k-10 \end{array}\right)$$

For unique solution: $\rho(A) = 3 = \rho(A|b)$. This is possible if $\lambda - 3 \neq 0$ or $\lambda \neq 3$.

For infinitely many solutions. $\rho(A) = \rho(A|b) < 3$

This is possible ~~if~~ $\lambda = 3$ & $k = 10$.

For No solution : $\rho(A) \neq \rho(A|b)$. This is possible if $\lambda = 3$ and $k \neq 10$.

3. $A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$. Eigenvalues of A are: $\lambda = -3, -1$

An eigenvector corresponding to $\lambda = -3$ is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

An eigenvector corresponding to $\lambda = -1$ is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Let $X = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \Rightarrow X^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

Define $Z = X^{-1}Y$ or $Y = XZ$. Then

$$Y' = AY + b \Rightarrow Z' = DZ + h, \text{ where}$$

$$D = \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix}, \; h = X^{-1}b = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2e^{-x} \\ 3x \end{pmatrix}$$

$$z_j' = \lambda_j z_j + h_j \quad , \; j = 1, 2$$

$$\therefore z_j = c_j e^{\lambda_j x} + e^{\lambda_j x}\int e^{-\lambda_j x} h_j(x)\, dx \quad , \; j = 1, 2$$

$$z_1 = c_1 e^{-3x} + \frac{1}{2} e^{-x} - \frac{1}{2}x + \frac{1}{6}$$

$$z_2 = c_2 e^{-x} + x e^{-x} + \frac{3}{2}x - \frac{3}{2}$$

$\therefore Y = XZ$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_1 + z_2 \\ -z_1 + z_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{-3x} + c_2 e^{-x} + x e^{-x} + \frac{1}{2} e^{-x} + x - \frac{4}{3} \\ -c_1 e^{-3x} + c_2 e^{-x} + x e^{-x} - \frac{1}{2} e^{-x} + 2x - \frac{5}{3} \end{pmatrix}.$$

4(a). $xy''+y'+y=0 \iff x^2y''+xy'+xy=0$

Let $y_1(x) = x^r \sum_{m=0}^{\infty} c_m x^m$, $c_0 \neq 0$. Substitute $y(x)$ and its derivatives in the given D.E. This gives

$$\sum_{m=0}^{\infty}\left\{\left[(m+r)(m+r-1)c_m + (m+r)c_m\right]x^{m+r} + c_m x^{m+r+1}\right\} = 0$$

$\Rightarrow (m+r)^2 c_m + c_{m-1} = 0$ (Coeff. of x^{m+r}, $c_{-1}=0$)

$m=0 \Rightarrow r^2=0$ (indicial equation) $\Rightarrow r=0$ is double root.

$\Rightarrow c_m = -\frac{1}{m^2} c_{m-1}$

$$\Rightarrow y_m = c_0 \sum_{m=0}^{\infty} \frac{(-1)^m x^m}{(m!)^2} = \sum_{m=0}^{\infty} \frac{(-1)^m x^m}{(m!)^2}, \quad (\text{with } c_0 = 1).$$

Since it is the case of double root, we have

$$y_2(x) = (\log x)\, y_1 + x^0 \sum_{m=1}^{\infty} b_m x^m .$$

Substitute $y_2(x)$ and its derivatives in the given D.E. This leads to

$$0 = 2xy_1' + \sum_{m=1}^{\infty}\left[b_m x^{m+1} + m b_m x^m + m(m-1) b_m x^m\right]$$

$$\Rightarrow \frac{2(-1)^m m}{(m!)^2} + b_{m-1} + m^2 b_m = 0, \quad m = 1, 3 \cdots$$

Where $b_0 = 0 \Rightarrow b_1 = 2$

$$\Rightarrow b_m = -\frac{b_{m-1}}{m^2} - \frac{2(-1)^m}{m\,(m!)^2}$$

$$\Rightarrow b_m = \frac{-2(-1)^m}{(m!)^2} h_m, \quad \text{where } h_m = 1 + \frac{1}{2} + \cdots + \frac{1}{m}.$$

$$\therefore y_2(x) = (\log x) \sum_{m=0}^{\infty} \frac{(-1)^m x^m}{(m!)^2} - 2 \sum_{m=1}^{\infty} \frac{(-1)^m}{(m!)^2} h_m x^m .$$

Hence the complete solution is given by

$$y(x) = c_0 \sum_{m=0}^{\infty} \frac{(-1)^m x^m}{(m!)^2} + c_1\left[\log x \sum_{m=0}^{\infty} \frac{(-1)^m x^m}{(m!)^2} - 2\sum_{m=1}^{\infty} \frac{(-1)^m h_m x^m}{(m!)^2}\right].$$

4(b). (i) $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$. (ii) $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$.

Let α, β be two consecutive positive zeros of $J_{n+1}(x)$. Using $\frac{d}{dx}\left[x^{n+1} J_{n+1}(x)\right] = x^{n+1} J_n(x$ (part (i) with $n = n+1$) By Rolle's theorem, there is a 'r' such that $\alpha < r < \beta$ and $J_n(r) = 0$. If there were more than one zero, say $\alpha < r < \delta < \beta$ using $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$ (part (ii)) Rolle's theorem would imply a zero of $J_{n+1}(x)$ in between r and δ, which would contradict consecutiveness of α and β. This completes the proof.

5(a). $((1+x^2)y')' + \lambda(x^2+1)^{-1}y = 0$, where prime denotes differentiation with respect to x.

$$\left((1+x^2)\frac{dy/dt}{dx/dt}\right)' + \lambda(x^2+1)^{-1}y = 0 \longrightarrow *$$

Substituting $\frac{dx}{dt} = x^2+1$

$$\int \frac{dx}{x^2+1} = dt \Rightarrow t = \arctan x \text{ or } x = \tan t$$

Substituting in $*$, $\Rightarrow \frac{d^2y}{dt^2} + \lambda y = 0$, whose general solution is

$$y = c_1 \cos\sqrt{\lambda}\, t + c_2 \sin\sqrt{\lambda}\, t$$

$0 = y(x=0) = y(t=0) \Rightarrow c_1 = 0$, therefore $y = c_2 \sin\sqrt{\lambda}\, t$

$0 = y(x=1) = y(t=\frac{\pi}{4}) \Rightarrow \sin\sqrt{\lambda}\frac{\pi}{4} = 0 = \sin n\pi$ (for nontrivial solution $c_2 \neq 0$)

Therefore, the eigenvalues are $\lambda_n = 16n^2$

and the corresponding eigenfunctions are $y_n = \sin(4nt)$
$= \sin(4n \arctan x)$,

$n = 1, 2, 3, \cdots$

5(b). Let $\phi_i(x) = \cos\frac{i\pi x}{L}$, $\phi_j(x) = \cos\frac{j\pi x}{L}$

For $i \neq j$, Consider $\langle \phi_i(x), \phi_j(x)\rangle = \int_{-L}^{L} \phi_i(x)\phi_j(x)\,dx = \int_{-L}^{L} \cos\frac{i\pi x}{L}\cos\frac{j\pi x}{L}\,dx = 0$

This shows that the given set of functions are orthogonal

Now, the norm of $\phi_i = \|\cos\frac{i\pi x}{L}\| = \langle \phi_i(x), \phi_i(x)\rangle^{1/2} = \left(\int_{-L}^{L}\cos^2\frac{i\pi x}{L}\,dx\right)^{1/2} = L^{1/2}$
$= \sqrt{L}$

$\therefore$ Orthonormal set is $\frac{\phi_i}{\|\phi_i\|} = \left\{\frac{1}{\sqrt{L}}\cos\frac{i\pi x}{L}\right\}_{i=1}^{\infty}$

For $i=0$, $\|\phi_0\| = \langle \phi_0(x), \phi_0(x)\rangle^{1/2} = \left(\int_{-L}^{L} dx\right)^{1/2} = \sqrt{2L}$

$\therefore$ Orthonormal set is $\left\{\frac{1}{\sqrt{2L}}, \frac{1}{\sqrt{L}}\cos\frac{i\pi x}{L}\right\}_{i=1}^{\infty}$

6(a). Prove existence of Laplace transform

6(b). $\mathcal{L}\{t^{-\frac{1}{2}}\} = \frac{\Gamma(\frac{1}{2})}{s^{\frac{1}{2}}} = \sqrt{\frac{\pi}{s}}$; $\mathcal{L}\{e^{-t}t^{-\frac{1}{2}}\} = \sqrt{\frac{\pi}{s+1}}$

$$\mathcal{L}^{-1}\left\{\frac{1}{s\sqrt{s+1}}\right\} = \frac{1}{\sqrt{\pi}}\int_0^t e^{-\tau}\tau^{-\frac{1}{2}}\,d\tau = \frac{1}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-x^2}x^{-1}\,2x\,dx = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-x^2}dx = \operatorname{erf}(\sqrt{t}).$$

$\left(\because \frac{F(s)}{s} = \mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\}\right)$