ELL333 Minor Exam 1 Solutions

1. Find e^{At} when

(a) $A = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix}$, $\lambda_1, \lambda_2 \in \mathbb{R}$

(b) A = any square matrix of your choice except the one in (a) above.

(a) $$A = \left[\begin{array}{cc|cc} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ \hline 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{array}\right] \equiv \left[\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right]$$

Let us partition matrix...

$$\Rightarrow e^{At} = \begin{bmatrix} e^{A_1 t} & e^{0t} \\ e^{0t} & e^{A_2 t} \end{bmatrix} = \begin{bmatrix} e^{A_1 t} & I \\ I & e^{A_2 t} \end{bmatrix}.$$

Now, $e^{A_2 t} = \begin{bmatrix} e^{\lambda_2 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$; it is a diagonal.

This is a scaling.

$$e^{A_1 t} = I + A_1 t + A_1^2 \frac{t^2}{2!} + \cdots$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \lambda_1 t & t \\ 0 & \lambda_1 t \end{bmatrix} + \begin{bmatrix} \lambda_1 \frac{t^2}{2!} & 2\lambda_1 \frac{t^2}{2!} \\ 0 & \lambda_1 \frac{t^2}{2!} \end{bmatrix} + \cdots$$

$$= \begin{bmatrix} e^{\lambda_1 t} & t e^{\lambda_1 t} \\ 0 & e^{\lambda_1 t} \end{bmatrix}$$

$t e^{\lambda_1 t}$ $(\lambda_1 < 0)$

$\Rightarrow$ a pulse.

- Jordan Form
- called transient growth / non-normal growth.

normal ~ 90°

Recall that Jordan Form emerges when two eigenvectors join. (Problem Set 2)

(b) $$A = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$$

This is a rotation.

$$A^2 = \begin{bmatrix} -\omega^2 & 0 \\ 0 & -\omega^2 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 0 & \omega^3 \\ -\omega^3 & 0 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} \omega^4 & 0 \\ 0 & \omega^4 \end{bmatrix}$$

$$\Rightarrow e^{At} = \begin{bmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{bmatrix}$$

2. Show that $\frac{d}{dt}\{\det(e^{At})\} = \text{trace}(A)\{\det(e^{At})\}$, where $A \in \mathbb{R}^{n\times n}$ is diagonalisable.

- If $A \in \mathbb{R}^{2\times 2}$ was diagonal $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$,

$$e^{At} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$$

$$\det(e^{At}) = e^{\lambda_1 t}.e^{\lambda_2 t} = e^{(\lambda_1+\lambda_2)t} = e^{\text{trace}(A).t}.$$

$$\Rightarrow \frac{d}{dt}\det(e^{At}) = \text{trace}(A).e^{\text{trace}(A).t}$$

$$= \text{trace}(A).\det(e^{At}).$$

- Above steps extend to an $n\times n$ diagonal matrix.

- If A was diagonalisable, there is an invertible matrix T such that $A = TDT^{-1}$, where D is diagonal.

$$\Rightarrow e^{At} = Te^{Dt}T^{-1}$$

$$\Rightarrow \det(e^{At}) = \det(Te^{Dt}T^{-1})$$

$$= \det(T).\det(e^{Dt}).\det(T^{-1})$$

$$= \det(T).\det(e^{Dt}).\frac{1}{\det(T)}$$

may be visualised through the volume interp-retation of the determinant

$$= \det(e^{Dt}) = e^{\text{trace}(D)t} = e^{\text{trace}(A)t}$$

see below

the det-trace formula.

$$\Rightarrow \frac{d}{dt}\det(e^{At}) = \frac{d}{dt}(\det(e^{Dt}))$$

$$= \text{trace}(D)\det(e^{Dt})$$

$$= \text{trace}(T^{-1}AT)\det(e^{At})$$

this is because the trace is a sum of diagonal elements

$$= \text{trace}(A\underbrace{T.T^{-1}}_{I})\det(e^{At})$$

$$= \text{trace}(A).\det(e^{At}).$$

- Any square matrix can be expressed as $A = TJT^{-1}$, where J is the Jordan Form. Above steps should work in this case.

3. Consider $\dot{x} = Ax$, $x \in \mathbb{R}^{2\times 1}$ and $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{22} & a_{22} \end{bmatrix}$,

where $a_{11}, a_{12}, a_{22}, a_{21} \in \mathbb{R}$. $\cdot$ trace $(A) \neq 0$ $\cdot$ det $(A) \neq 0$

(a) Find a real symmetric matrix $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$, that satisfies $A^T P + PA = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$p_{11} = ?$, $p_{12} = ?$, $p_{22} = ?$

(b) What are the conditions on $\{a_{11}, a_{12}, a_{21}, a_{22}\}$ for P to be positive definite?

[P is positive definite if $x^T P x > 0 \ \forall x \neq 0$].

(a) $A^T P = \begin{bmatrix} a_{11}p_{11} + a_{21}p_{11} & a_{11}p_{12} + a_{21}p_{22} \\ a_{12}p_{11} + a_{22}p_{12} & a_{12}p_{12} + a_{22}p_{22} \end{bmatrix}$, $PA = (A^T P)^T$

$\therefore A^T P + PA = -I \Rightarrow a_{11}p_{11} + a_{21}p_{12} = -1/2$

$a_{12}p_{11} + a_{22}p_{12} + a_{11}p_{12} + a_{21}p_{22} = 0$

$a_{12}p_{12} + a_{22}p_{22} = -1/2$

$$\Rightarrow \underbrace{\begin{bmatrix} a_{11} & a_{21} & 0 \\ a_{12} & a_{11}+a_{22} & a_{21} \\ 0 & a_{12} & a_{22} \end{bmatrix}}_{\text{det is trace}(A)\times\det(A) \neq 0} \begin{bmatrix} p_{11} \\ p_{12} \\ p_{22} \end{bmatrix} = \begin{bmatrix} -1/2 \\ 0 \\ -1/2 \end{bmatrix} \Rightarrow \begin{bmatrix} p_{11} \\ p_{12} \\ p_{22} \end{bmatrix} = \frac{-1/2}{\text{trace}(A) \times \det(A)} \begin{bmatrix} \det(A) + a_{22}^2 + a_{12}^2 \\ -a_{21}a_{22} - a_{11}a_{12} \\ \det(A) + a_{11}^2 + a_{21}^2 \end{bmatrix}$$

(b) $x^T P x = p_{11}x_1^2 + 2p_{12}x_1x_2 + p_{22}x_2^2$

If $p_{11} = 0$, $x^T P x = x_2(2p_{12}x_1 + p_{22}x_2)$ which could be > 0 or < 0.

So $p_{11} \neq 0$, $x^T P x = p_{11}\left(x_1^2 + 2x_1 \frac{p_{12}}{p_{11}} x_2 + \left(\frac{p_{12}}{p_{11}}x_2\right)^2 - \left(\frac{p_{12}}{p_{11}}x_2\right)^2\right) + p_{22}x_2^2$

$= p_{11}\left(x_1 - \frac{p_{12}}{p_{11}}x_2\right)^2 + \left(\frac{p_{11}p_{22} - p_{12}^2}{p_{11}}\right)x_2^2$ = sum of squares

> 0 if $p_{11} > 0$ and $p_{11}p_{22} - p_{12}^2 > 0$. (Sylvester's criterion for $n=2$)

$\Rightarrow \frac{-1}{\text{trace}(A).\det(A)} . (\det(A) + a_{22}^2 + a_{12}^2) > 0$ $\Rightarrow$ trace $(A) < 0$

and $\det(A) . (\underbrace{(a_{11}+a_{22})^2 + (a_{12}-a_{21})^2}_{\text{positive}}) > 0$ $\Rightarrow$ det $(A) > 0$

(some calculation)

4. Consider the system $\dot{x} = 2x + u$.

(a) For $u = 0$, is the point $x = 0$ stable. Why or why not?

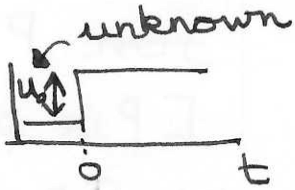
(b) For $x(0) = 0$, design the input u such that $x(1) = 10$.

(a) No. eigenvalue $= 2 > 0$

(b) Let's solve

$$\dot{x} = 2x + u$$

For the class of step inputs*



$$\frac{d}{dt} x(t) e^{-2t} = u_0 e^{-2t}, \quad t \geq 0$$

$$\Rightarrow x(t) e^{-2t} = \frac{u_0}{-2}(e^{-2t} - 1)$$

$$\Rightarrow x(t) = \frac{u_0}{2}(e^{2t} - 1)$$

For $x(1) = 10$, $\quad u_0 = \dfrac{20}{e^2 - 1}$.

* Other inputs possible

→ Note that trace $(A) < 0$ and $\det(A) > 0$ is the condition for the characteristic polynom-ial of A, $s^2 - \underset{ace}{tr}(A)\, s + \det(A)$ to have eigenval-ues with negative real parts. This is what is expected from the Theorem in class that relates the positive definiteness of P with negative real parts of the eigenvalues of A.