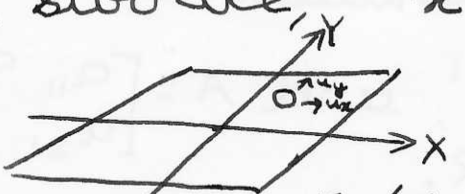
ELL333 Minor Exam 2 1 hr 19 marks

1. $A \in \mathbb{R}^{n\times n}$ is a square matrix. Show that the matrix exponential can be expressed as a finite sum,

$$e^{At} = \phi_0(t) I + \phi_1(t) A + \dots + \phi_{n-1}(t) A^{n-1} \quad \forall t \geq 0 .$$

2. Show that
(A,B) is controllable $\Longleftrightarrow$ rank $[A-\lambda I \;\; B] = n$ for all complex numbers λ.

3. Consider a model of a ball rolling on a surface, $\ddot{x} = u_x$, $\ddot{y} = u_y$. Design the input $u = \begin{bmatrix} u_x \\ u_y \end{bmatrix}$ so that the ball approaches the state $\cancel{\begin{bmatrix} x(0) \\ \dot{x}(0) \\ y \end{bmatrix}}$ $\begin{bmatrix} x \\ \dot{x} \\ y \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ starting from the state $\begin{bmatrix} x \\ \dot{x} \\ y \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. Discuss your design choices, such as the form of u and any gain~~s~~ values.



4. Consider $\dot{x} = Ax + Bu$, $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(a) Is the system controllable? Justify.

(b) Does there exist a function $u(t)$ so that $x(1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ starting from $x(0) = \begin{bmatrix} e^{-4} \\ e^{-3} \end{bmatrix}$? Elaborate on your answer.

Q.	Marks
1	3
2	6
3	5
4	2+3

ELL333 Minor Exam 2 Solutions

1. $A \in \mathbb{R}^{n \times n}$ is a square matrix. Show that the matrix exponential can be expressed as a finite sum,

$$e^{At} = \phi_0(t) I + \phi_1(t) A + \ldots + \phi_{n-1}(t) A^{n-1} \quad \forall\, t \geq 0.$$

The Cayley-Hamilton Theorem implies that A satisfies its own characteristic polynomial,

$$A^n + \alpha_1 A^{n-1} + \alpha_2 A^{n-2} + \ldots + \alpha_n I = 0,$$

where $\lambda^n + \alpha_1 \lambda^{n-1} + \alpha_2 \lambda^{n-2} + \ldots + \alpha_n \equiv \det(\lambda I - A)$.

$$\Rightarrow A^n = -\alpha_1 A^{n-1} - \alpha_2 A^{n-2} \ldots - \alpha_n I. \quad \text{—①}$$

$$e^{At} = I + tA + \frac{t^2}{2!} A^2 + \ldots, \quad \text{—②}$$

by definition.

Each term A^k, $k \geq n$ may be expressed in terms of $I, A, \ldots A^{n-1}$ by repeated application of Eqn. ①.

From Eqn. ②, the co-efficients are some functions of time.

$$\therefore e^{At} = \phi_0(t) I + \phi_1(t) A + \ldots + \phi_{n-1}(t) A^{n-1}, \ \forall\, t \geq 0$$

as desired.

This is an example where (countable) infinite is reduced to finite (via the Cayley-Hamilton Theorem).

2. Show that

(A, B) is controllable $\Longleftrightarrow$ rank$\{[A-\lambda I \;\; B]\}$ $= n$ for all complex numbers λ.

($\Rightarrow$) Suppose

rank$\{[A-\lambda I \;\; B]\} < n$ for a λ.

$\Rightarrow \exists\, v (\neq 0)$ such that

$v^* [A-\lambda I \;\; B] = 0$,

$\Rightarrow [v^*(A-\lambda I) \;\; v^* B] = 0$,

$\Rightarrow v^*(A-\lambda I) = 0, \; v^* B = 0$,

$\Rightarrow v^* A = \lambda v^*, \; v^* B = 0$.

Consider $v^* [B \; AB \ldots A^{n-1}B]$,

$= [v^* B \;\; v^* AB \ldots v^* A^{n-1} B]$,

$= [0 \;\; \lambda v^* B \ldots \lambda v^* \Lambda^{n-2} B]$,

...

$= [0 \;\; 0 \ldots 0]$,

$\Rightarrow$ rank$\{[B \; AB \ldots A^{n-1} B]\} < n$,

$\Rightarrow$ (A, B) is not controllable.

This is a contradiction.

($\Leftarrow$) Suppose (A, B) is not controllable.

$\Rightarrow$ rank$\{[B \; AB \; A^2B \ldots A^{n-1}B]\} = r < n$

$\Rightarrow \exists\, T$ such that

$$T^{-1}AT = \tilde{A} = \left[\begin{array}{c:c} \tilde{A}_{11} & \tilde{A}_{12} \\ \hdashline 0 & \tilde{A}_{22} \end{array}\right],$$

$$T^{-1}B = \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix} = \tilde{B}.$$

$$[\tilde{A}-\lambda I \quad \tilde{B}] = [T^{-1}AT - T^{-1}(\lambda I)T \quad T^{-1}B]$$

$$= T^{-1}[AT - \lambda I T \quad B]$$

$$= T^{-1}[A-\lambda I \quad B]\begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix}$$

→ same rank, because they are related by invertible transformations.

$$\therefore \text{rank}\{[A-\lambda I \quad B]\}$$

$$= \text{rank}\{[\tilde{A}-\lambda I \quad \tilde{B}]\}$$

$$= \text{rank}\left\{\left[\begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix} - \lambda I \quad \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix}\right]\right\}$$

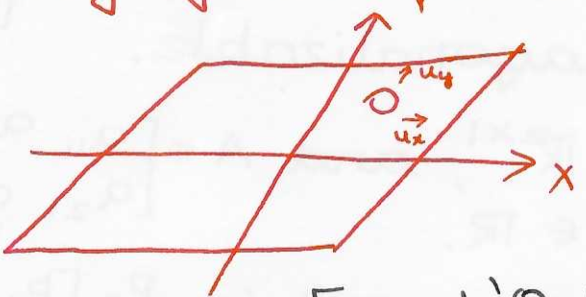
$$= \text{rank}\left\{\begin{bmatrix} \tilde{A}_{11}-\lambda I & \tilde{A}_{12} & \tilde{B}_1 \\ 0 & \tilde{A}_{22}-\lambda I & 0 \end{bmatrix}\right\}$$

$< n$ when λ is an eigenvalue of $\tilde{A}_{22}$.

"Often a student studying mathematics has difficulty in understanding why or how a particular result, or method of proof, was ever conceived in the first place. Sometimes ideas seem to appear from nowhere. Now it is true that mathematical geniuses do invent radically new results, and methods for proving old results, which often appear quite strange. The most that ordinary people can do is to accept these brilliant ideas for what they are, try to understand their consequences, and build on them to obtain further information."

– Earl A. Coddington

3. Consider a model of a ball rolling on a surface, $\ddot{x} = u_x$, $\ddot{y} = u_y$. Design the input $u = \begin{bmatrix} u_x \\ u_y \end{bmatrix}$ so that the ball approaches the state $\begin{bmatrix} x \\ \dot{x} \\ y \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ starting from the state $\begin{bmatrix} x \\ \dot{x} \\ y \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. Discuss your design choices such as the form of u and any gain value.



$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \\ y \\ \dot{y} \end{bmatrix} = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} x \\ \dot{x} \\ y \\ \dot{y} \end{bmatrix} + \left[\begin{array}{c|c} 0 & 0 \\ 1 & 0 \\ \hline 0 & 0 \\ 0 & 1 \end{array}\right] \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

- Effectively, two identical decoupled system, each with $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
- This is controllable.
- Choose $u_x = -k_1 x - k_2 \dot{x}$.

Choice of linear control law motivated by simplicity.

- $A - BK = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix}$

$= \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}$ whose eigenvalues are roots of $\lambda^2 + k_2 \lambda + k_1 = 0$.

Choose any $k_1, k_2 > 0$ and we are guaranteed that $x \to 0$, $\dot{x} \to 0$ as $t \to \infty$ because $\mathrm{Re}\{A - BK\} < 0$.
(Eigenvalue)

This is what is desired.

Large values of k_1, k_2 would speed the response at the expense of control effort.

Same for u_y.

"Initial training in pure and applied sciences tends to present problem-solving as the process of elaborating explicit closed-form solutions from basic principles, and then using these solutions in numerical applications. This approach is only applicable to very limited classes of problems... Unfortunately, most real-life problems are too complex to be amenable to this type of treatment."

- Éric Walter

4. Consider $\dot{x} = Ax + Bu$, $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(a) Is the system controllable? Justify.

(b) Does there exist a function $u(t)$ so that $x(1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ starting from $x(0) = \begin{bmatrix} e^{-4} \\ e^{-3} \end{bmatrix}$?

Elaborate on your answer.

(a) ~~Yes~~ No, because rank$\{[B \ AB]\}$

$= \text{rank}\left\{\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\right\} = 1 < 2$.

(b) Yes. Denote $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Then $x_2(t) = x_2(0)e^{3t}$

$\Rightarrow x_2(1) = e^{-3} \cdot e^{3} = 1$

x_1-dynamics are,

$\dot{x}_1 = 2x_1 + u$

For constant u, (after $t=0$)

$$x_1(t) = x_1(0)\, e^{2t} + e^{2t}\int_0^t u e^{-2\tau}\, d\tau,$$

$$\Rightarrow x_1(t) = e^{-4} \cdot e^{2t} + u e^{2t}\frac{(1 - e^{-2t})}{2},$$

$$\Rightarrow x_1(t) = e^{-4} \cdot e^{2t} + u\frac{(e^{2t} - 1)}{2}.$$

For $x_1(1) = 1$,

$$1 = e^{-4} \cdot e^{2} + u\frac{(e^{2} - 1)}{2},$$

and a u can be found! $\left\{\text{Even though not controllable}\right\}$

"... learn the fundamentals carefully and completely."

— S. C. Dutta Roy