ELL333 Problem Set 2

1. Consider Newton's Law of a body with no force, $m\ddot{x}=0$, $m=1$.

a) Using $x_1 = x$, $x_2 = \dot{x}$, write the model in the form $\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. $A = ?$

(b) Find the eigenvalues and eigenvectors of A.

(c) Find $x_1(t)$ and $x_2(t)$.

(d) What is $e^{At} = ?$

2. Find the eigenvalues and eigenvectors of (a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, (c) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, d) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(e) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, (f) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, (g) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, (h) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

For each of the above, find the matrix exponential.

3. Consider $A = \begin{bmatrix} \lambda & 1 \\ 0 & \mu \end{bmatrix}$. If $\lambda \neq \mu$, calculate the eigenvalues and eigenvectors of A as well as e^{At}. What is $\lim_{\mu \neq \to \lambda} e^{At}$? Sketch the eigenvectors of A for $\mu = 10\lambda, 5\lambda, 2\lambda$.

4. Show that (a) $e^{0_{n\times n}} = I_{n\times n}$ (b) $\frac{d}{dt} e^{At} = A e^{At} = e^{At} \cdot A$

(c) $(e^{At})^{-1} = e^{-At}$ (d) $e^{A_1 + A_2} = e^{A_1} \cdot e^{A_2}$ only if $A_1 A_2 = A_2 A_1$,

(e) $e^{A(t+s)} = e^{At} \cdot e^{As}$

(f) $\frac{d}{dt}\{\det(e^{At})\} = \text{trace}(A) \{\det(e^{At})\}$

5. (Jordan Canonical Form) Consider a 2×2 matrix A that has a repeated eigenvalue λ, but only one eigenvector $v_1 \Rightarrow A v_1 = \lambda v_1$. Choose another vector v_2 that is linearly independent to v_1 to construct the basis $\mathcal{B} = \{v_1, v_2\}$.

(a) Can $A v_2 = c_1 v_1 + c_2 v_2$, for some scalars c_1 & c_2?

(b) Show that $A \underbrace{[v_1 \mid v_2]}_{T} = \underbrace{[v_1 \mid v_2]}_{T} \begin{bmatrix} \lambda & c_1 \\ 0 & c_2 \end{bmatrix}$.

(c) The representation of A in the basis $\mathcal{B}$ is $\begin{bmatrix} \lambda & c_1 \\ 0 & c_2 \end{bmatrix}$. What values of c_1 and c_2 would render this the simplest representation, in your opinion?

(d) For $c_1 = 1$, $c_2 = \lambda$, show that the vector v_2 would have to satisfy $(A - \lambda I)^2 v_2 = 0$ and $(A - \lambda I) v_2 \neq 0$. What would happen if $(A - \lambda I) v_1 = 0$?

(e) For $J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$, find e^{Jt}.

(f) What would change in the above steps if $A \in \mathbb{R}^{3 \times 3}$, λ was repeated 3 times, but had only one eigenvector.

[Note: v_2 in (d) is called a generalised eigenvector]

6. For $\dot{x} = a x$, $x \in \mathbb{R}$, assume a series solution $x(t) = b_0 + b_1 t + b_2 t^2 + b_3 t^3 + \dots$
Find $b_0, b_1, \dots$ by substituting.