Problem Set 3

1. Investigate the stability of $\dot{x} = -x^3$ using the Lyapunov Function candidate $V(x) = x^4$.

2. Consider $\dot{x} = Ax$, $x \in \mathbb{R}^{2\times 1}$, and $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ where $a_{ij} \in \mathbb{R}$, $i,j = 1,2$. Find a real symmetric matrix P that satisfies $A^T P + PA = -I$ (2x2 identity). What are the conditions on the a_{ij}'s for P to be positive definite? How do these compare with the conditions that the eigenvalues of A have a negative real part.

3. A model (linearized) of a bicycle,
$M\ddot{q} + vC_1\dot{q} + (v^2 K_2 + gK_0)q = f$, $q = \begin{bmatrix} \phi \\ \delta \end{bmatrix}$; $f = \begin{bmatrix} T_\phi \\ T_\delta \end{bmatrix}$
ϕ: balance angle, δ: steer angle,
Set $\phi = x_1$, $x_2 = \delta$, $x_3 = \dot{\phi}$, $x_4 = \dot{\delta}$.

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline \multicolumn{2}{c|}{-M^{-1}(gK_0 + v^2 K_2)} & \multicolumn{2}{c}{-M^{-1}vC_1} \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Parameters: $M = \begin{bmatrix} 80.8 & 2.3 \\ 2.3 & 0.3 \end{bmatrix}$, $C_1 = \begin{bmatrix} 0 & 33.9 \\ -0.9 & 1.7 \end{bmatrix}$, $K_0 = \begin{bmatrix} -81 & -2.6 \\ -2.6 & -0.8 \end{bmatrix}$
$g = 9.8$, $v = 2$, $K_2 = \begin{bmatrix} 0 & 76.6 \\ 0 & 2.7 \end{bmatrix}$

Is the system stable?
What would happen for different values of forward velocity, v?
{Watch the TEDx Talk "Why bicycles do not fall" by A. Schwab}.

4. Consider the system,

$$\dot{x}_1 = 2x_1 + 3x_2 + 2x_3 + x_4$$
$$\dot{x}_2 = -2x_1 - 3x_2$$
$$\dot{x}_3 = -2x_1 - 2x_2 - 4x_3$$
$$\dot{x}_4 = -2x_1 - 2x_2 - 2x_3 - 5x_4$$

Write this in the form $\dot{x} = Ax$.

Convert A into a diagonal form.
{This example is due to Kalman.}

5. For $\dot{z} = z + u$, $z(0) = 0$, design an input u so that $z(1) = 10$.

6. The Sylvester equation is $AX + XB = C$, where $A \in \mathbb{C}^{m\times m}$, $B \in \mathbb{C}^{n\times n}$, $C \in \mathbb{C}^{m\times n}$ are known, and $X \in \mathbb{C}^{m\times n}$ is unknown. Show that $\int_0^\infty e^{At} C e^{Bt}\, dt \,(= X)$ is a solution, provided the integral exists.