ELL 333 Problem Set 4

1. Determine the controllability of

$$\dot{x}_1 = 2x_1 + 3x_2 + 2x_3 + x_4 + u$$
$$\dot{x}_2 = -2x_1 - 3x_2 - 2u$$
$$\dot{x}_3 = -2x_1 - 2x_2 - 4x_3 + 2u$$
$$\dot{x}_4 = -2x_1 - 2x_2 - 2x_3 - 5x_4 - u.$$

Find the controllable subspace. Find a similarity transformation that transforms this system into a form where the controllable and uncontrollable states are separated.

2. Determine the controllability of

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline \multicolumn{2}{c|}{-M^{-1}\times} & \multicolumn{2}{c}{-M^{-1}\times} \\ \multicolumn{2}{c|}{[gK_0 + v^2 K_2]} & \multicolumn{2}{c}{v\, C_1} \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \hline \multicolumn{2}{c}{M^{-1}} \end{array}\right] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

where $M = \begin{bmatrix} 80.8 & 2.3 \\ 2.3 & 0.3 \end{bmatrix}$, $g = 9.8$, $v = 2$

$C_1 = \begin{bmatrix} 0 & 33.9 \\ -0.9 & 1.7 \end{bmatrix}$, $K_0 = \begin{bmatrix} -81 & -2.6 \\ -2.6 & -0.8 \end{bmatrix}$, $K_2 = \begin{bmatrix} 0 & 76.6 \\ 0 & 2.7 \end{bmatrix}$.

3. Is the following system controllable?

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\Omega^2 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u.$$

Can its eigenvalues be arbitrarily placed? Using $u = -Kx$? Are there unique values of K for which eigenvalues are at $-1, -2$?

4. Verify the controllability of

$$A_c = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_n \\ 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

5. Show that
(A, B) is controllable $\Leftrightarrow$
rank $[A-\lambda I \mid B] = n$, for all complex numbers λ.

ELL 333 Problem set 5

1. Use $y = 7x_1 + 6x_2 + 4x_3 + 2x_4$ in Q1 of Problem Set 4. Is this observable? Find the observable subspace. Find a similarity transformation that transforms this system into a form where the observable and unobservable states are separated.

2. For the bicycle model in Q2 of Problem Set 4, determine the observability if the measured output is
a) x_1, b) x_2, c) x_3, d) $x_1 - x_2$.

3. Show that
(C, A) is observable $\Leftrightarrow$
rank $\begin{bmatrix} A-\lambda I \\ C \end{bmatrix} = n$, for all complex numbers λ.

4. Suppose rank $\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = r < n$, then show that there exists a similarity transformation T such that
$$TAT^{-1} = \begin{bmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}; \quad CT^{-1} = [\tilde{C}_1 \ \ 0]$$ and that
$(\tilde{C}_1, \tilde{A}_{11})$ is observable.

5. Show that a similarity transformation doesnot change observability

6. A model of an inverted pendulum (length, l) is

$$\frac{d}{dt}\begin{bmatrix}\theta\\ \dot\theta\end{bmatrix} = \begin{bmatrix}\dot\theta\\ \frac{2l}{1+l^2}\theta + \frac{1}{2}u\end{bmatrix}$$ input

Output $y = \theta$.

It is desired to balance this in the inverted position.

a) Using $x = \begin{bmatrix}\theta\\ \dot\theta\end{bmatrix}$, represent the model in the form $\dot x = Ax + Bu$, $y = Cx + Du$. Find A, B, C, D.

b) Analyse the stability.

c) How would the ease of balancing depend on the length, l?

d) Show that the system is controllable.

e) Design a state feedback control $u = -kx$ so that the eigenvalues of the closed-loop system are at the roots of $s^2 + \frac{5}{2}s + \frac{3}{2} = 0$.

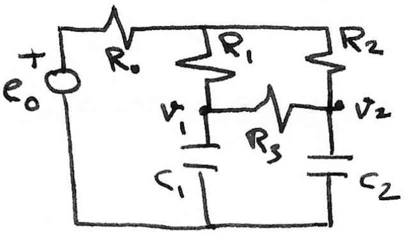
f) Calculate the transfer functions $C(sI - A)^{-1}$ and $C(sI - (A - BK))^{-1}$.

g) Show that the system is observable.

h) Design an observer so that the observer eigenvalues are at the roots of the quadratic equation $s^2 + 2s + 2 = 0$.

i) Write down the equations of the overall controller - observer.

7. Is the bridge circuit



$$\dot v_1 = -\frac{1}{C_1}\left(\frac{1}{R_1} + \frac{1}{R_3}\right)v_1 + \frac{1}{C_1R_3}v_2 + \frac{1}{C_1R_1}e_0$$

$$\dot v_2 = \frac{1}{C_2R_3}v_1 - \frac{1}{C_2}\left(\frac{1}{R_2} + \frac{1}{R_3}\right)v_2 + \frac{1}{C_2R_2}e_0$$

observable if $y = v_1 - v_2$?

8. Consider the system $\dot{x} = Ax + Bu$, $y = Cx$,

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

a) Show that (A, B) is controllable. Design a controller $u = -kx$ to place eigenvalues of closed-loop system at the roots of the equation $s^2 + 2\zeta_c\omega_c s + \omega_c^2 = 0$, $\zeta_c, \omega_c > 0$.

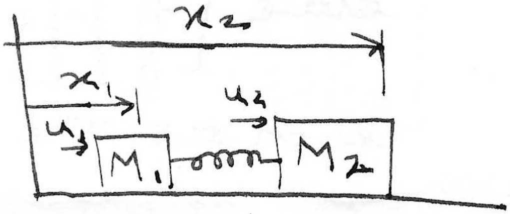
b) Show that (A, C) is observable. Design an observer with gain L so that the eigenvalues of $A - LC$ are at the roots of the equation $s^2 + 2\zeta_o\omega_o s + \omega_o^2 = 0$, $\zeta_o, \omega_o > 0$.

c) What factors might govern the choice of $\{\zeta_c, \omega_c, \zeta_o, \omega_o\}$?

9. The equations of motion of a pair of masses M_1 and M_2 coupled by a spring and sliding in one dimension in the absence of friction are

$$\ddot{x}_1 + \frac{K}{M_1}(x_1 - x_2) = \frac{u_1}{M_1},$$

$$\ddot{x}_2 + \frac{K}{M_2}(x_2 - x_1) = \frac{u_2}{M_2},$$



where u_1 and u_2 are externally applied forces and K is the spring constant. Suppose $y = [x_1 \ x_2]^T$. Investigate the controllability and observability of the system.

10. A simplified model of "balancing a stick on the hand" when the hand does not translate is

$$\dot{\theta} = \omega,$$

$$\dot{\omega} = \Omega^2\theta - \alpha\omega + u, \quad \alpha > 0.$$

a) Design a stabilising controller $u = -K\begin{bmatrix} \theta \\ \omega \end{bmatrix}$.

b) Design an observer to obtain state estimate $\{\hat{\theta}, \hat{\omega}\}$ from measurements of θ.