IMPEDANCE

BY A. E. KENNELLY

The impedance of a conductor is its apparent resistance, and is expressible in ohms. More strictly, it has been defined as the ratio of the effective E.M.F. between the terminals of the conductor, to the effective current strength it carries. By voltage or current in the sequel, effective voltage and effective current, unless otherwise specified, will be intended,—such as would be indicated by properly calibrated alternating current voltmeters and ammeters.

With steady continuous currents, the impedance of a conductor is its simple resistance. With periodically fluctuating currents, the impedance will in general differ from the resistance, and will be greater or less, but in common practice greater. With such voltages and currents, Ohm's law becomes

$$C = \frac{E}{I}$$

where the impedance is represented by I.

We may first consider periodic currents and voltages of the simple harmonic type, following waves in time such as could be traced upon a band of paper, moving steadily lengthwise, by the vertical shadow of the crank-pin on a fly-wheel revolving uniformly in a vertical plane perpendicular to the band. [Fig. 1.] The ratio of the velocity between wheel and paper would control the spacing and steepness of the waves, but would not alter their sinusoidal type, while the motion of the shadow would be simply harmonic.

If the fly-wheel makes n complete periods or revolutions per second, and the distance of the crank-pin from the shaft axis —its crank radius—is unity, then the circumference of the circle it traces will be 2π, and the total distance it will run through in one second will be $2\pi n$ or $6.283n$ linear units, which we may call its speed, and denote by $p = 2\pi n$. If then the pin is shifted outwards upon the wheel until its crank-radius or distance from the axis is l units, where l is the number of henrys in any particular inductance, then the corresponding travel of the pin will be $2\pi ln$ linear units per second, and this speed pl we may call the inductance-speed for the particular frequency and inductance l considered.

Non-ferric inductances (if the term be permitted), or inductances without iron may be first assumed, leaving ferric inductances and conductors embracing iron, for later examination.

In order to find the impedance of any conductor whose resistance is r ohms, and inductance l henrys, we construct the triangle having as base the length r, the perpendicular of pl units, the draw the hypothenuse which will have the length $\sqrt{r^2 + (pl)^2}$ between their free extremities. [See Fig. 1.] This length will be the impedance in ohms. The impedance is therefore the geometrical or vector[1] sum of the resistance and inductance-speed, when these are plotted on two rectangular axes. Calling this impedance i, Ohm's law gives

$$c = \frac{e}{i} \qquad e = ic \qquad i = \frac{e}{c}$$

corresponding to the usual formulas for continuous currents. In Fig. 2 a resistance of 75 ω having an inductance of 0.06 henry with a consequent inductance speed at 100 ~ , of

$$6.283 \times 100 \times 0.06 = 37.7$$

is shown to possess an impedance i of

$$\sqrt{(75)^2 + (37.7)^2} = 83.9 \text{ ohms.}$$

If two such impedances, i_1 and i_2, be connected in series, their total impedance [Fig. 3] will be the vector sum AC of i_1 and i_2 corresponding to the arithmetical sum of unvarying currents. This vector sum is most conveniently constructed geometrically by summing separately the component resistances r_1, r_2, AG, and then the component inductance-speeds, GC, as shown. Calling the vector sum I,

the voltage on I for a current of c amperes will be cI volts.
" " " i_1 " " " " " " ci_1 "
" " " i_2 " " " " " " ci_2 "

but I will not be the arithmetical sum of i_1 and i_2, nor will the drop on I be the sum of the drops of i_1 and i_2, unless the separate time constants, l_1/r_1 and l_2/r_2 are equal, when AB will be in line with BC; the reason being that otherwise the currents in i_1 and i_2 while of equal strength, are out of step.

In Fig. 3, r_1 represents to scale, a resistance of 30 ω, r_2 20 ω, pl_1 an inductance speed of 10, pl_2 30. Then

$$i_1 \text{ is } \sqrt{900 + 100} = 31.63 \text{ ohms,}$$

$$i_2 \text{ is } \sqrt{400 + 900} = 36.06 \text{ ohms,}$$

$$I = \sqrt{2500 + 1600} = 64.0 \text{ ohms,}$$

whereas the arithmetical sum of i_1 and i_2 would be 67.69 ohms.

A condenser of capacity k farads (millions of microfarads) has an impedance of $1/pk$ ohms, or in other words, its impedance is the reciprocal of its capacity speed.[2] The current in a condenser supplied by an E.M.F. e at n ~ is

$$e \div \frac{1}{pk} = epk.$$

Thus, 1000 volts alternating give through 5 microfarads at 100 ~

$$1000 \times 6.283 \times 100 \times \frac{5}{1{,}000{,}000} = 3.14 \text{ amperes.}$$

A condenser k in series with a resistance r has a total impedance

$$\sqrt{r^2 + \left(\frac{1}{pk}\right)^2}$$

[1] The vector sum of two lines AB and BC is the line AC, since a transference from A to B followed by a transference from B to C is geometrically equivalent to a transference from A to C.

[2] Fleming, "The Alternate Current Transformer," Vol. II., p. 378.

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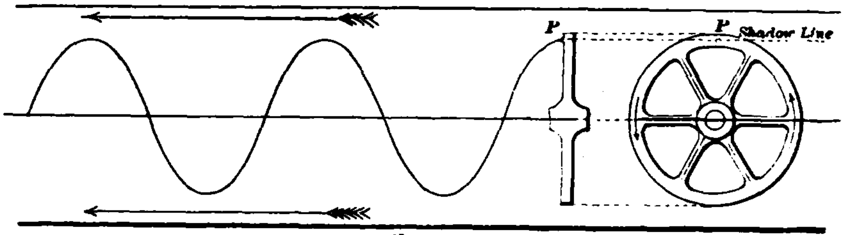


FIG. 1.

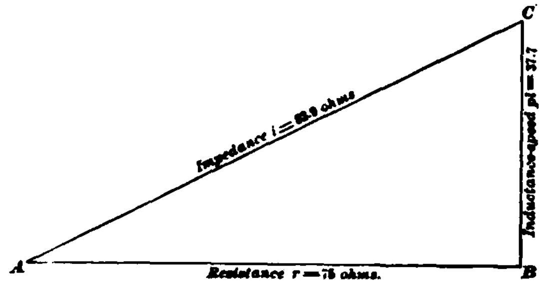


FIG. 2.

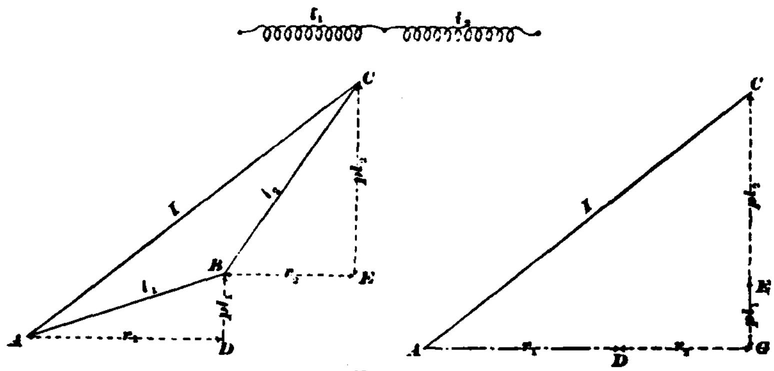


FIG. 3.

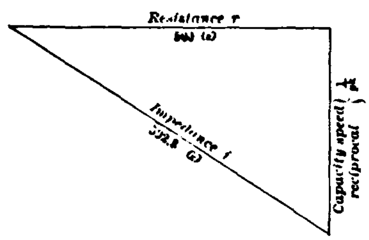


FIG. 4.

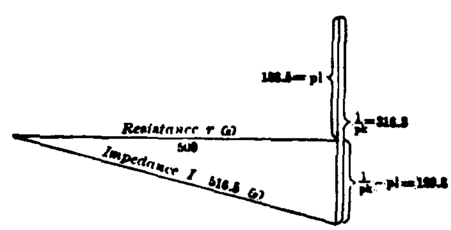


FIG. 5.

which is found by taking the base r, and the capacity-speed reciprocal $1/pk$ downwards perpendicularly. In other words, its impedance is the vector sum of the resistance and capacity-speed reciprocal.

Fig. 4 represents the five-microfarad-condenser above considered, whose impedance $1/pk$ at 100 ~ is 318.3 ohms, in circuit with a non-inductive resistance r of 500 ohms. The resulting impedance

$$I = \sqrt{500^2 + 318.3^2} = 592.8 \text{ ohms},$$

so that 1000 volts alternating harmonically at that frequency would, when connected to this series deliver

$$\frac{1000}{592.8} = 1.687 \text{ amperes through the same.}$$

If again, inductance is inserted in series with the condenser and resistance, the total impedance of the three, independent of their order, will be the vector sum of the resistance, the inductance-speed, and the capacity-speed-reciprocal, the second being drawn upwards, and the third negatively or downwards, as shown in Fig. 5, where an inductance of 300 millihenrys = 0.3 henry, giving at 100 ~ an inductance-speed of

$$628.3 \times 0.3 = 188.5,$$

is introduced into the same circuit of resistance and condenser. The effect is to diminish the impedance from 592.8 to 516.6 ohms, increasing the current to 1.936 amperes. If I be the resulting impedance,

$$c = \frac{e}{I},$$

and the voltage across the resistance and inductance supposing these combined in one coil will be ci_1 where i_1 is the impedance of these two considered together, and the voltage across the condenser will be

$$c \cdot \frac{1}{pk}.$$

It is evident that this latter may be greater than e when suitable values are chosen so that one or both of the factors, current and capacity-speed-reciprocal, are large. In such cases the condenser voltage exceeds the voltage of supply, and what is commonly called the "Ferranti effect," is developed.

It is also evident from the construction of the diagrams, that the inductance will neutralize the capacity in its series or *vice-versa*, when the verticals above and below are equal, or when the inductance-speed equals the capacity-speed-reciprocal, and in this case the impedance degrades into the simple resistance. This condition requires that

$$pl = \frac{1}{kp}, \quad \text{or } l = \frac{1}{kp^2}, \quad \text{or } k = \frac{1}{lp^2}.$$

With unvarying currents, the combined resistance of resistances in parallel, is the reciprocal of the sum of their reciprocals. Thus, if three resistances be connected in multiple, of 3, 2, and 6 ohms respectively, their joint resistance is

$$\frac{1}{\frac{1}{3}+\frac{1}{2}+\frac{1}{6}} = 1.$$

In the same way, the joint impedance of impedances in multiple, is the reciprocal of the vector sum of their reciprocals. Thus suppose that three impedances, i_1, i_2, i_3, are connected in parallel, the first being, for example, a simple resistance; the second a series of resistance, inductance, and capacity; the third a simple condenser. These three are diagramatically represented in Fig. 6. Take the numerical reciprocals of i_1, i_2, and i_3, and lay them off in their respective directions. Take the vector sum of these, which is most conveniently done by resolving the reciprocals into components, summing rs alone the pls alone. Finally the reciprocal of this vector sum, laid off in its direction, will be the joint impedance of the combination, and the components of this impedance will represent the equivalent resistance, and the equivalent capacity-speed-reciprocal or equivalent inductance-speed of the combination.

Thus, in Fig. 6, $i_1 = r_1 = 400$ ω non-inductive: i_2 is the series of inductance, resistance, and capacity, taken from the last case represented in Fig. 5, and i_3 is a simple condenser of 5 microfarads or 0.000005 farads whose capacity speed will be $628.3 \times 0.000005 = 0.0031416$, and whose impedance, the reciprocal of this, is 318.3 ohms: i_1, i_2, and i_3 are represented to scale in their proper directions, AB $= i_1$ horizontal, CE $= i_2$ diagonal, and GH $= i_3$ vertically downwards. Immediately beneath each of these are their respective vector reciprocals,

$$\text{A'B'} = 0.0025, \quad \text{C'E'} = 0.001936, \quad \text{and G'H'} = 0.0031416,$$

maintaining their directions, but to an altered dimensional scale for convenience in working. The vector sum of these reciprocals, A'B', C'E' and G'H', is A'H', indicated in the quadrilateral 1 and measures 0.005682. The most convenient way to obtain this practically is, however, shown at triangle 2, where the projected horizontal component of C'E', namely CD 0.001874, is added to A'B', and also the projected vertical component D'E' 0.00048, to G'H'. The hypothenuse of the triangle, A'H', then measures 0.005682 either arithmetically or geometrically. The vector reciprocal of A'H' is AH, represented at triangle 3, and measures 1/0.005682 or 176 ohms on the original scale, with projected components AM 135.5 ω and MH capacity-speed-reciprocal 112.3.

If the voltage measured between terminals O and P is 1000 volts, the current in the main leads to O and P is

$$\frac{1000}{176} = 5.682 \text{ amperes.}$$

But the current in the branch i_1 is $\dfrac{1000}{i_1} = \dfrac{1000}{400} = 2.5$ amperes

Also " " " " " i_2 " $\dfrac{1000}{i_2} = \dfrac{1000}{516.6} = 1.936$ "

And " " " " " i_3 " $\dfrac{1000}{i_3} = \dfrac{1000}{318.3} = 3.142$ "

Sum, 7.578

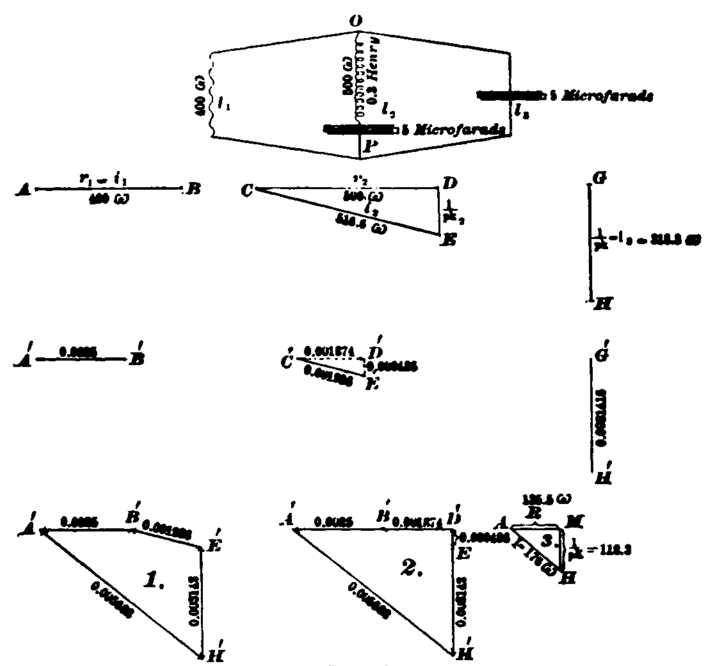


Fig. 6.

The sum of the branch currents is therefore greater in this instance than the current supplied through the main, for the reason that these currents are not in step; but at any one instant the main current and the branch currents' sum must be equal; it is only in the process of averaging over the cycle that the difference manifests itself.

If two conductors are connected in parallel, as for example a galvanometer and its shunt, of resistances g and s respectively, we know that if C is the unvarying current supplied through the combination, the portion through one of them,—the galvanometer say—will be $C(s/g+s)$. In the same way, if G and S are their respective impedances to a harmonic current, the galvanometer's share will be $C(S/G+S)$, where, however, $G+S$ is not the arithmetical sum, but the vector sum of the two impedances. Either G and S may contain capacity, provided that this is included in its impedance by the usual rule of laying off the capacity-speed-reciprocal against the inductance-speed.

Again, the joint resistance formula for galvanometer and shunt with steady currents being

$$r=\frac{gs}{g+s},$$

the corresponding formula is

$$i=\frac{GS}{G+S},$$

where GS is the arithmetical product, but $G+S$ a vector sum. This case could also be dealt with by the rule for joint impedances

$$i=\frac{1}{\frac{1}{G}+\frac{1}{S}},$$

where the vector reciprocals are added vectorially.

All continuous current corollaries from Ohm's law thus become applicable to harmonic currents, when the sums and differences of impedances or their reciprocals are treated vectorially, and if all alternating currents were strict sinusoids, all inductances non-ferric, there would be almost as little difficulty in reckoning their quantitative relations as in continuous current circuits. But Kirchoff's laws, while true instantaneously, are only applicable to alternating current circuits, when the currents are all in step, and the vector sums degrade into arithmetical sums.

Algebraically, all, and more than all the foregoing statements concerning impedance combinations are contained in the following:

Any combination of resistances, non-ferric inductances, and capacities, carrying harmonically alternating currents, may be treated by the rules of unvarying currents, if the inductances are considered as resistances of the form $pl\sqrt{-1}$, and the capacities as resistances of the form $-(1/kp)\sqrt{-1}$, the algebraic operations being then performed according to the laws controlling "complex quantities."

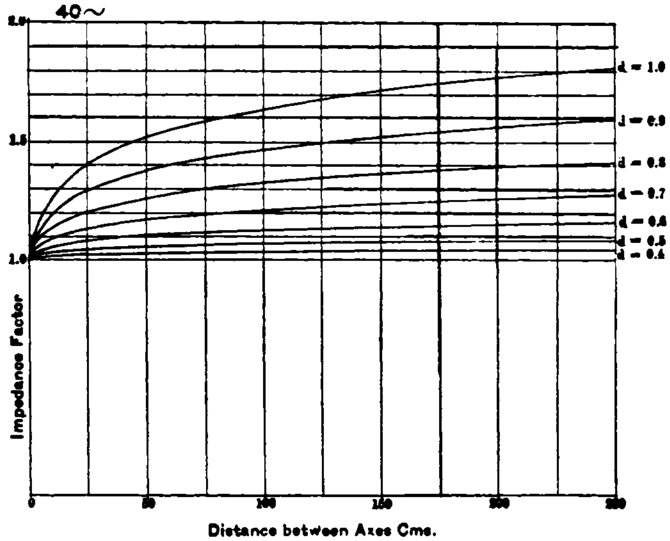
A valuable application of the principles of impedance is to the determination of the limits of error in measuring apparatus—such as voltmeters—intended for both alternating and continuous current circuits.

For example, a particular sample of alternating and continuous current voltmeter, consisting of a dynamometer without iron, in circuit with a resistance coil, has a resistance of 2,085 ohms, and an inductance that varies with the relative positive of the dynamometer coils, but which does not exceed 91.5 millihenrys = 0.0915 henry. At a frequency of 140 ~ which is about the maximum in ordinary use, the inductance speed of this instrument will not be greater than $0.0915\times 6.283\times 140=80.5$. Constructing the right-angled triangle with base 2085, and perpendicular 80.5, we find the hypotenuse to be $\sqrt{2085^2+80.5^2}=2086.5$ which is the impedance of the instrument at this frequency, so that 50 volts E.M.F. harmonically alternating would force less current through the dynamometer than 50 volts continuous E.M.F. in the ratio of 2085/2086.5, and would therefore under-indicate by about 0.075 per cent.

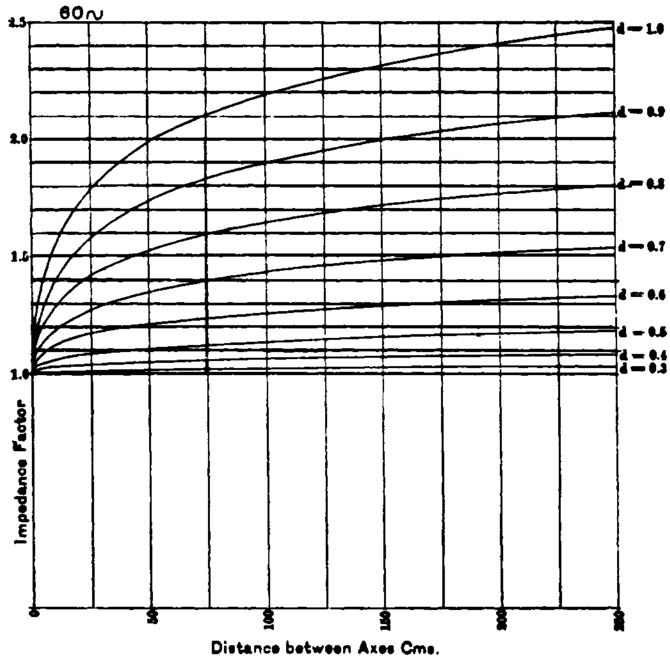
A similar instance is afforded by the Thomson wattmeter. The armature of this instrument has a certain non-ferric inductance and is connected across the supply mains through a resistance. The equality of the meter's record for both alternating and continuous current circuits depends upon the equality of current strength through the armature under equal alternating and continuous voltages, and this is again dependent on the impedance of the armature circuit. In a particular instrument of 1,500 watts capacity (50 volts and 30 amperes) the total resistance of this circuit is 455.5 ohms, and its total inductance 25 millihenrys (0.025 henry). This represents at 140 ~ an inductance-speed of 22, and an impedance of 456.0 ohms $=\sqrt{(455.5)^2+(22)^2}$ so that the current in the armature will be about one-ninth of one per cent. less, on a supply pressure of 50 volts, alternating harmonically, than on 50 volts continuous. Strictly speaking the accuracy of a wattmeter depends not only upon the non-diminution of current strength through fine wire circuit in virtue of inductance, but also upon the lag in phase that is also there established. It is generally safe to assume, however, that when the error due to impedance is very small, the error due to lag is very small also.

An important application of impedance principles is to conductors for the distribution of alternating currents. Let the common case be considered of two parallel overhead wires running from a central station to a consumption point one mile distant, each wire of copper, No. 4, A.W.G., 0.2043 in. (0.519 cm.) in diameter, fixed on pole insulators, at a distance between their axes of 30 inches. At a temperature of 15°C this wire of standard conductivity, would have a resistance of 1.285 ohms per mile of 5,280 feet, and if the wire actually possessed this conductivity, the resistance of the loop would be 2.57 ohms. An unvarying current of 10 amperes would establish in this wire a "drop" of 12.85 volts on each conductor, or 25.7 volts in the circuit. But if 10 amperes are supplied harmonically at a frequency of 120 ~ , the "drop" will no longer be 12.85 volts on each wire, but 19.27, or almost 50 per cent. more, making the total "drop" in the loop 38.54 volts. The impedance of the loop is one and one-half times its resistance, or the impedance factor is in this case 1.5. The additional drop in the wires is owing to the inductance of the loop. The magnetic flux from the current moves back and forth through this loop at each pulse of the current, and establishes a back pressure or counter E.M.F. that compounds with the ordinary cr drop, rectangularly. No attention is here paid to the static capacity of the line, which is indeed of minor importance in overhead wires of only a few miles in length, but whose influence on the impedance factor increases with the capacity of the line, the frequency, and the E.M.F.

In order to reduce the impedance of our pair of overhead wires as far as possible, we must bring them close together, so that the loop may hold less flux, and the counter E.M.F. from



Curve-Plate I.



Curve-Plate II.

the oscillation of this, will be correspondingly lessened. A safe practical distance might be ten inches (25.4 cms.) apart. By bringing them to this distance their impedance would be reduced to 34.7 ohms for the whole loop, and the impedance factor from 1.5 to 1.35. At this distance of ten inches between wire axes, the impedance of the circuit of No. 4 A.W.G. will be a little less than with No. 3, A.W.G., at 30 in., although No. 3 has 26 per cent. more weight and cross-section. The energy expended thermally in the wires as c^2r, is of course strictly in proportion to the ohmic resistance, so that in respect to loss of energy, No. 3 would have the full advantage of its weight, but as far as regards drop in voltage, with its attendant influence upon the pressure regulation over the system, No. 3 at 30 in. is almost the exact equivalent of No. 4 at 10 in.

If the frequency had been 140 ~ in place of 120 ~ , the impedance factors of 1.5 and 1.35 would have become 1.9 and 1.45 respectively, and the Nos. 4 wires at 10 in. would have had 4 per cent. less impedance and drop, than Nos. 3 at 2 ft. 6 in. apart.

It is evident that for alternating current supply circuits which commonly consist of pairs of overhead line wires, the influence of their proximity is of practical importance, and grows with the size of wire. They should clearly be placed for lowest impedance at the minimum intervening distance consistent with safety; on adjacent pins at the same side of the pole, rather than on opposite sides. Not only is their impedance thus reduced, but their inductive disturbing influence upon parallel telegraph or telephone circuits is reduced also. A still better means of reducing the impedance is to subdivide the lines, and employ several loops of small wire rather than one loop of large wire.

Curve sheets I to VIII, corresponding to Tables I to VIII are arranged to indicate by direct inspection the impedance factors of such pairs of overhead copper wires. The covering of

TABLE I.

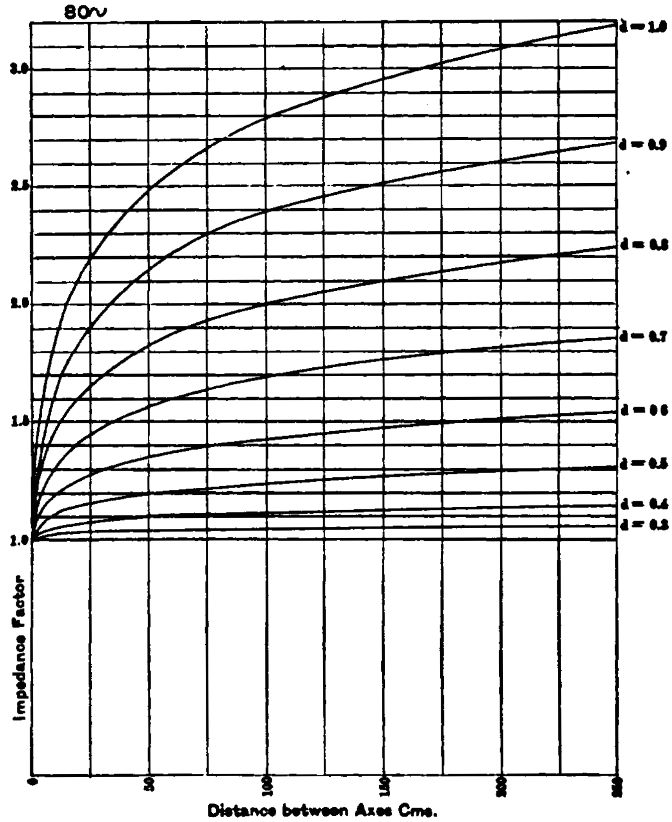
IMPEDANCE FACTORS FOR PAIRS OF PARALLEL COPPER WIRES CARRYING SIMPLE HARMONIC CURRENTS OF 140 ~

$\frac{D.}{r}$	Diam. 1 m.m. Rad. 0.5 m.m. D. cms.	factor.	Diam. 0.2 cm. Rad. 0.1 cm. D. cms.	factor.	Diam. 0.3 cm. Rad. 0.15 cm. D. cms.	factor.	Diam. 0.4 cm. Rad. 0.2 cm. D. cms.	factor.	Diam. 0.5 cm. Rad. 0.25 cm. D. cms.	factor.	Diam. 0.6 cm. Rad. 0.3 cm. D. cms.	factor.	Diam. 0.7 cm. Rad. 0.35 cm. D. cms.	factor.	Diam. 0.8 cm. Rad. 0.4 cm. D. cms.	factor.	Diam. 0.9 cm. Rad. 0.45 cm. D. cms.	factor.	Diam. 1.0 cm. Rad. 0.5 cm. D. cms.	factor.
2	0.1	1.00003	0.2	1.0005	0.3	1.002	0.4	1.0075	0.5	1.019	0.6	1.038	0.7	1.069	0.8	1.115	0.9	1.179	1.	1.263
3							0.6	1.0155	0.75	1.037	0.9	1.076	1.05	1.137	1.2	1.224	1.35	1.341	1.5	1.489
4	0.2	1.00009	0.4	1.0014	0.6	1.007	0.8	1.023	1.0	1.055	1.2	1.110	1.4	1.196	1.6	1.317	1.8	1.475	2.	1.671
5							1.	1.029	1.25	1.070	1.5	1.140	1.75	1.247	2.	1.396	2.25	1.587	2.5	1.821
6	0.3	1.0001	0.6	1.0022	0.9	1.011	1.2	1.035	1.5	1.084	1.8	1.167	2.1	1.2925	2.4	1.464	2.7	1.683	3.	1.947
8							1.6	1.046	2.0	1.108	2.4	1.213	2.8	1.3683	3.2	1.577	3.6	1.839	4.	2.152
10	0.5	1.0002	1.0	1.0035	1.5	1.018	2.	1.055	2.5	1.128	3.	1.251	3.5	1.431	4.	1.669	4.5	1.965	5.	2.316
12							2.4	1.062	3.	1.146	3.6	1.284	4.2	1.484	4.8	1.746	5.4	2.070	6.	2.451
16	0.8	1.0003	1.6	1.005	2.4	1.024	3.2	1.076	4.	1.176	4.8	1.339	5.6	1.571	6.4	1.872	7.2	2.239	8.	2.668
20							4.	1.087	5.	1.200	6.	1.384	7.	1.641	8.	1.972	9.	2.372	10.	2.839
25	1.25	1.0004	2.5	1.0065	3.75	1.032	5.	1.098	6.25	1.226	7.5	1.430	8.75	1.713	10.	2.073	11.25	2.507	12.5	3.009
30							6.	1.109	7.5	1.248	9.0	1.469	10.5	1.773	12.	2.157	13.5	2.618	15.	3.151
40	2.0	1.0005	4.	1.0083	6.	1.041	8.	1.125	10.	1.284	12.	1.532	14.	1.869	16.	2.292	18.	2.796	20.	3.375
60							12.	1.150	15.	1.338	18.	1.624	21.	2.009	24.	2.485	27.	3.048	30.	3.693
80	4.0	1.0007	8.	1.0115	12.	1.057	16.	1.170	20.	1.378	24.	1.692	28.	2.109	32.	2.623	36.	3.229	40.	3.919
100							20.	1.185	25.	1.409	30.	1.745	35.	2.188	40.	2.732	45.	3.370	50.	4.096
150	7.5	1.001	15.	1.015	22.5	1.073	30.	1.214	37.5	1.469	45.	1.845	52.5	2.334	60.	2.930	67.5	3.627	75.	4.419
200							40.	1.236	50.	1.513	60.	1.916	70.	2.439	80.	3.073	90.	3.811	100.	4.648
300	15.	1.0012	30.	1.019	45.	1.092	60.	1.268	75.	1.576	90.	2.019	105.	2.588	120.	3.274	135.	4.071	150.	4.974
400							80.	1.291	100.	1.622	120.	2.093	140.	2.695	160.	3.418	180.	4.255	200.	5.202
500	25.	1.0014	50.	1.022	75.	1.108	100.	1.310	125.	1.658	150.	2.151	175.	2.778	200.	3.528	225.	4.40	250.	5.383
600							120.	1.325	150.	1.688	180.	2.199	210.	2.847	240.	3.622	270.	4.518	300.	5.530
700	35.	1.0016	70.	1.025	105.	1.119	140.	1.339	175.	1.713	210.	2.239	245.	2.904	280.	3.699	315.	4.617	350.	5.654
800							160.	1.351	200.	1.736	240.	2.274	280.	2.955	320.	3.766	360.	4.704	400.	5.675
900	45.	1.0017	90.	1.026	135.	1.127	180.	1.362	225.	1.756	270.	2.306	315.	2.999	360.	3.826	405.	4.780	450.	5.858
1,000							200.	1.370	250.	1.773	300.	2.333	350.	3.039	400.	3.879	450.	4.849	500.	5.942
1,500	75.	1.0019	150.	1.030	225.	1.145	300.	1.407	375.	1.842	450.	2.442	525.	3.193	600.	4.084	675.	5.112	750.	6.269
2,000					300.	1.155	400.	1.434	500.	1.892	600.	2.519	700.	3.303	800.	4.231	900.	5.299	1000.	6.502
3,000	150.	1.0023	300.	1.036	450.	1.170	600.	1.472	750.	1.963	900.	2.630	1050.	3.458	1200.	4.437	1350.	5.563	1500.	6.831
4,000					600.	1.181	800.	1.492	1000.	2.013	1200.	2.707	1400.	3.566	1600.	4.581	1800.	5.747	2000.	7.059
5,000	250.	1.0026	500.	1.040	750.	1.190	1000.	1.522	1250.	2.053	1500.	2.769	1750.	3.655	2000.	4.698	2250.	5.897	2500.	7.244
10,000			1000.	1.047	1500.	1.220	2000.	1.595	2500.	2.183	3000.	2.968	3500.	3.933	4000.	5.067	4500.	6.366	5000.	7.827
15,000			1500.	1.051	2250.	1.236	3000.	1.633	3750.	2.252	4500.	3.073	5250.	4.079	6000.	5.261	6750.	6.615	7500.	8.134
20,000	1000.	1.0034	2000.	1.054	3000.	1.249	4000.	1.664	5000.	2.305	6000.	3.153	7000.	4.222	8000.	5.409	9000.	6.802	10000.	8.366

TABLE II.

IMPEDANCE FACTORS FOR PAIRS OF PARALLEL COPPER WIRES CARRYING SIMPLE HARMONIC CURRENTS OF 120 ~ AT 15°C.

$\frac{D.}{r}$	Diam. 0.1 cm. Rad. 0.05 cm. D.		Diam. 0.2 cm. Rad. 0.1 cm. D.		Diam. 0.3 cm. Rad. 0.15 cm. D.		Diam. 0.4 cm. Rad. 0.2 cm. D.		Diam. 0.5 cm. Rad. 0.25 cm. D.		Diam. 0.6 cm. Rad. 0.3 cm. D.		Diam. 0.7 cm. Rad. 0.35 cm. D.		Diam. 0.8 cm. Rad. 0.4 cm. D.		Diam. 0.9 cm. Rad. 0.45 cm. D.		Diam. 1.0 cm. Rad. 0.5 cm D.	
	cms.	factor.	cms.	factor.	cms.	factor.	cms.	factor.	cms.	factor.	cms.	factor.	cms.	factor.	cms.	factor.	cms.	factor.	cms.	factor.
2	0.1	1.00002	0.2	1.0004	0.3	1.002	0.4	1.0056	0.5	1.0135	0.6	1.028	0.7	1.051	0.8	1.086	0.9	1.134	1.	1.199
3							0.6	1.0114	0.75	1.028	0.9	1.057	1.05	1.102	1.2	1.169	1.35	1.260	1.5	1.3765
4	0.2	1.00006	0.4	1.001	0.6	1.005	0.8	1.017	1.	1.040	1.2	1.082	1.4	1.147	1.6	1.241	1.8	1.365	2.	1.522
5							1.	1.022	1.25	1.052	1.5	1.105	1.75	1.187	2.	1.303	2.25	1.455	2.5	1.644
6	0.3	1.0001	0.6	1.002	0.9	1.008	1.2	1.026	1.5	1.062	1.8	1.125	2.1	1.222	2.4	1.357	2.70	1.532	3.	1.747
8							1.6	1.034	2.	1.080	2.4	1.160	2.8	1.281	3.2	1.447	3.6	1.659	4.	1.916
10	0.5	1.00017	1.	1.003	1.5	1.013	2.	1.041	2.5	1.100	3.	1.190	3.5	1.330	4.	1.521	4.5	1.761	5.	2.051
12							2.4	1.046	3.	1.109	3.6	1.215	4.2	1.372	4.8	1.583	5.4	1.848	6.	2.163
16	0.8	1.00023	1.6	1.004	2.4	1.018	3.2	1.056	4.	1.132	4.8	1.258	5.6	1.442	6.4	1.685	7.2	1.987	8.	2.344
20							4.	1.064	5.	1.151	6.	1.293	7.	1.498	8.	1.767	9.	2.098	10.	2.487
25	1.25	1.003	2.5	1.005	3.75	1.024	5.	1.073	6.25	1.170	7.5	1.329	8.75	1.556	10.	1.851	11.25	2.210	12.5	2.631
30							6.	1.081	7.5	1.187	9.	1.360	10.5	1.604	12.	1.920	13.5	2.303	15.	2.750
40	2.	1.00038	4.	1.006	6.	1.031	8.	1.093	10.	1.215	12.	1.410	14.	1.683	16.	2.031	18.	2.451	20.	2.938
60							12.	1.113	15.	1.257	18.	1.484	21.	1.797	24.	2.192	27.	2.663	30.	3.207
80	4.	1.00053	8.	1.008	12.	1.042	16.	1.127	20.	1.288	24.	1.539	28.	1.880	32.	2.307	36.	2.815	40.	3.399
100							20.	1.13	25.	1.313	30.	1.582	35.	1.945	40.	2.398	45.	2.934	50.	3.550
150	7.5	1.00068	15.	1.011	22.5	1.050	30.	1.161	37.5	1.360	45.	1.663	52.5	2.066	60.	2.564	67.5	3.152	75.	3.822
200							40.	1.178	50.	1.395	60.	1.721	70.	2.153	80.	2.683	90.	3.306	100.	4.018
300	15.	1.00087	30.	1.014	45.	1.069	60.	1.203	75.	1.445	90.	1.806	105.	2.278	120.	2.854	135.	3.527	150.	4.294
400							80.	1.221	100.	1.482	120.	1.866	140.	2.366	160.	2.975	180.	3.684	200.	4.490
500	25.	1.0010	50.	1.016	75.	1.080	100.	1.235	125.	1.511	150.	1.915	175.	2.437	200.	3.070	225.	3.807	250.	4.643
600							120.	1.247	150.	1.535	180.	1.954	210.	2.493	240.	3.147	270.	3.907	300.	4.769
700	35.	1.0011	70.	1.018	105.	1.088	140.	1.258	175.	1.556	210.	1.987	245.	2.542	280.	3.212	315.	3.991	350.	4.875
800							160.	1.267	200.	1.574	240.	2.017	280.	2.585	320.	3.270	360.	4.066	400.	4.967
900	45.	1.0012	90.	1.020	135.	1.095	180.	1.275	225.	1.590	270.	2.043	315.	2.622	360.	3.322	405.	4.130	450.	5.049
1,000							200.	1.283	250.	1.605	300.	2.065	350.	2.656	400.	3.364	450.	4.188	500.	5.120
1,500	75.	1.0014	150.	1.023	225.	1.108	300.	1.312	375.	1.660	450.	2.156	525.	2.785	600.	3.540	675.	4.413	750.	5.400
2,000							400.	1.333	500.	1.701	600.	2.220	700.	2.878	800.	3.663	900.	4.572	1000.	5.598
3,000	150.	1.0017	300.	1.026	450.	2.128	600.	1.363	750.	1.759	900.	2.312	1050.	3.009	1200.	3.839	1350.	4.797	1500.	5.878
4,000							800.	1.385	1000.	1.801	1200.	2.378	1400.	3.102	1600.	3.964	1800.	4.956	2000.	6.077
5,000	250.	1.0019	500.	1.030	750.	1.143	1000.	1.403	1250.	1.833	1500.	2.429	1750.	3.174	2000.	4.061	2250.	5.080	2500.	6.231
10,000					1500.	1.166	2000.	1.460	2500.	1.940	3000.	2.596	3500.	3.410	4000.	4.374	4500.	5.482	5000.	6.730
15,000	750.	1.0024	1500.	1.037	2250.	1.178	3000.	1.492	3750.	1.997	4500.	2.684	5250.	3.536	6000.	4.540	6750.	5.694	7500.	6.992
20,000	1000.	1.0025	2000.	1.040	3000.	1.188	4000.	1.516	5000.	2.041	6000.	2.752	7000.	3.630	8000.	4.666	9000.	5.854	10000.	7.191



Curve Plate III.

the wire, if free from iron, of course plays no part, or at least no practically appreciable part, in this matter. The tables are based upon Matthiessen's standard conductivity, and a temperature of 15°C (59°F), representing perhaps the mean annual temperature of wires in the temperate zone of climate. The factors are strictly speaking only correct at this temperature, but if applied to the resistance of a wire at its actual temperature, the error is usually small. Thus the impedance factor for a No. 5 A.W.G. wire at 9.1 in. (23.1 cm.) interaxial distance, is seen to be 1.311 at 140 ~ , so that its resistance at standard conductivity and 15°C of 0.0003069 ohms per foot becomes an impedance under these conditions of 0.0004024 ohms per foot. But if the wires were at the freezing point of water, their resistance per foot at that temperature would be 0.0002897 ohms, an impedance with the same factor of 0.0003798, while the correct value for temperature 0°C would be 0.0003895 and the corrected factor 1.344, a discrepancy of about 2.5 per cent. for a change of temperature amounting to 15°C. A correction formula, for a greater degree of accuracy, is, however, given in Appendix I.

Tables and curves I to VI inclusive are of general application, dealing with wires from 1 mm. up to 1 cm. in diameter, and also all interaxial distances up to 250 cms., for the successive frequencies of 40, 60, 80, 100, 120 and 140 ~ . Tables and curves VII and VIII apply more particularly to American practice, dealing with all sizes of A.W.G. wires from No. 000 to No. 11 inclusive, and all interaxial distances up to 100 inches for the two frequencies 120 ~ and 140 ~ . Factors for intermediate sizes or frequencies can readily be found by interpolation. For instance, suppose that the impedance factor is required for a pair of No. 6 A.W.G. wires suspended at an interaxial distance of 40 inches at 15°C, and at a simply harmonic frequency of 130 ~ . Curves VII and VIII give

No. 6 at 140 ~ = 1.34 at 40 in.
" " 120 ~ = 1.26 " "

Taking the mean value for the midway frequency, we have No. 6 at 130 ~ = 1.30. Direct calculation yields in fact 1.300 as the factor at 130 ~ .

The evidence upon which these tables and curves are based is not merely theoretical. Observations made upon voltage drops on overhead wires in actual alternating current systems have coroborated the results. A particular set of observations is also added in Appendix I, together with a simple development of the formula from which the tables have been computed to the fourth significant digit.

The same tables and curves will of course apply to secondary circuits or interior wiring, and even if the pairs of wires are twisted together into double cords, without serious error, the only exception being to wires laid in, or close to iron pipes.

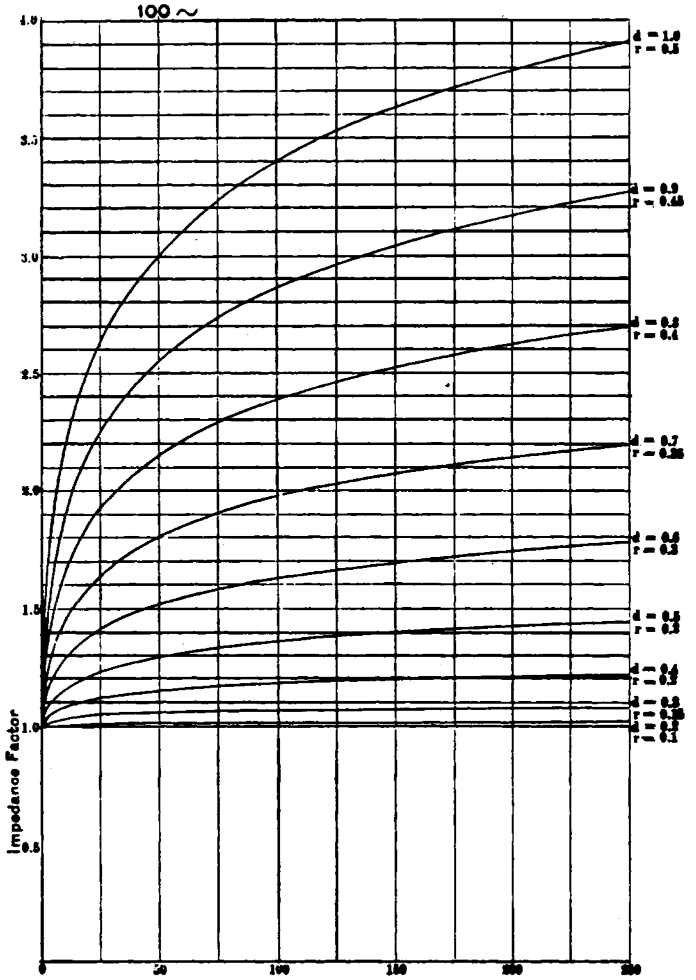
Single copper wires suspended overhead at a mean elevation h, with ground return, have an impedance as though that ground return were replaced by an insulated copper wire of equal size vertically beneath, and at a distance h below the surface. An elevation h, thus represents a loop interaxial distance of $2h$, and the imaginary wire below ground is the "image" of the suspended wire, so called from the optical analogy with respect to the surface midway between. Heaviside has pointed out[3] that this result is based upon the assumption that the return current through the ground spreads uniformly through a thin film over the surface, and does not

[3]"Reprint of Electrical Papers," vol. i, p. 101.

TABLE III.

IMPEDANCE FACTORS FOR PAIRS OF PARALLEL COPPER WIRES CARRYING SIMPLE HARMONIC CURRENTS OF 100 ~ AT 15°C.

$\frac{D.}{r}$	Diam. 0.1 cm. Rad. 0.05 cm. *D.*		Diam. 0.2 cm. Rad. 0.10 cm. *D.*		Diam. 0.3 cm. Rad. 0.15 cm. *D.*		Diam. 0.4 cm. Rad. 0.2 cm. *D.*		Diam. 0.5 cm. Rad. 0.25 cm. *D.*		Diam. 0.6 cm. Rad. 0.3 cm. *D.*		Diam. 0.7 cm. Rad. 0.35 cm. *D.*		Diam 0.8 cm. Rad. 0.4 cm. *D.*		Diam. 0.9 cm. Rad. 0.45 cm. *D.*		Diam. 1.0 cm. Rad. 0.5 cm. *D.*	
	cms.	factor.	cms.	factor.	cms.	factor.	cms.	factor.	cms.	factor.	cms.	factor.	cms.	factor.	cms.	factor.	cms.	factor.	cms.	factor.
2			0.2	1.0002	0.3	1.0012	0.4	1.004	0.5	1.010	0.6	1.020	0.7	1.036	0.8	1.061	0.9	1.095	1.0	1.142
3							0.6	1.008	0.75	1.019	0.9	1.040	1.05	1.072	1.2	1.120	1.35	1.187	1.5	1.274
4			0.4	1.0007	0.6	1.004	0.8	1.012	1.	1.028	1.2	1.058	1.4	1.104	1.6	1.172	1.8	1.265	2.	1.384
5							1.	1.015	1.25	1.036	1.5	1.074	1.75	1.133	2.0	1.218	2.25	1.332	2.5	1.477
6			0.6	1.0011	0.9	1.006	1.2	1.018	1.5	1.044	1.8	1.088	2.1	1.158	2.4	1.258	2.7	1.391	3.	1.557
8							1.6	1.024	2.0	1.056	2.4	1.114	2.8	1.202	3.2	1.326	3.6	1.489	4.	1.689
10			1.0	1.0018	1.5	1.009	2.	1.028	2.5	1.067	3.	1.135	3.5	1.239	4.0	1.383	4.5	1.569	5.	1.796
12							2.4	1.032	3.	1.077	3.6	1.154	4.2	1.279	4.8	1.431	5.4	1.636	6.	1.886
16			1.6	1.0025	2.4	1.013	3.2	1.039	4.	1.093	4.8	1.185	5.6	1.323	6.4	1.509	7.2	1.746	8.	2.030
20							4.	1.045	5.	1.107	6.	1.211	7.	1.366	8.	1.573	9.	1.834	10.	2.145
25			2.5	1.003	3.75	1.017	5.	1.051	6.25	1.121	7.5	1.238	8.75	1.410	10.	1.638	11.25	1.923	12.5	2.261
30							6.	1.057	7.5	1.133	9.	1.270	10.5	1.447	12.	1.693	13.5	1.997	15.	2.357
40			4.	1.004	6.	1.021	8.	1.066	10.	1.154	12.	1.299	14.	1.508	16.	1.781	18.	2.116	20.	2.510
60							12.	1.080	15.	1.184	18.	1.355	21.	1.596	24.	1.908	27.	2.287	30.	2.720
80			8.	1.006	12.	1.030	16.	1.090	20.	1.207	24.	1.396	28.	1.661	32.	2.001	36.	2.410	40.	2.886
100							20.	1.098	25.	1.226	30.	1.430	35.	1.712	40.	2.073	45.	2.507	50.	3.009
150	7.5	1.0005	15.	1.008	22.5	1.038	30.	1.115	37.5	1.261	45.	1.492	52.5	1.809	60.	2.208	67.5	2.684	75.	3.233
200							40.	1.127	50.	1.287	60.	1.537	70.	1.877	80.	2.304	90.	2.810	100.	3.394
300			30.	1.010	45.	1.048	60.	1.145	75.	1.325	90.	1.603	105.	1.977	120.	2.442	135.	2.991	150.	3.621
400	20.	1.0007					80.	1.158	100.	1.353	120.	1.651	140.	2.048	160.	2.540	180.	3.120	200.	3.783
500			50.	1.011	75.	1.056	100.	1.169	125.	1.376	150.	1.689	175.	2.104	200.	2.617	225.	3.220	250.	3.909
600							120.	1.178	150.	1.394	180.	1.719	210.	2.150	240.	2.679	270.	3.302	300.	4.011
700	35.	1.0008	70.	1.013	105.	1.062	140.	1.185	175.	1.410	210.	1.746	245.	2.189	280.	2.734	315.	3.371	350.	4.099
800							160.	1.192	200.	1.423	240.	1.769	280.	2.224	320.	2.780	360.	3.432	400.	4.176
900			90.	1.014	135.	1.067	180.	1.198	225.	1.436	270.	1.790	315.	2.254	360.	2.822	405.	3.486	450.	4.243
1,000	50.	1.0009					200.	1.204	250.	1.447	300.	1.808	350.	2.281	400.	2.858	450.	3.533	500.	4.302
1,500			150.	1.016	225.	1.073	300.	1.225	375.	1.490	450.	1.880	525.	2.386	600.	3.001	675.	3.718	750.	4.533
2,000					300.	1.082	400.	1.241	500.	1.521	600.	1.931	700.	2.461	800.	3.103	900.	3.850	1000.	4.698
3,000	150.	1.0012	300.	1.019	450.	1.090	600.	1.263	750.	1.566	900.	2.004	1050.	2.568	1200.	3.246	1350.	4.035	1500.	4.929
4,000					600.	1.096	800.	1.280	1000.	1.599	1200.	2.057	1400.	2.643	1600.	3.348	1800.	4.167	2000.	5.093
5,000	250.	1.0013	500.	1.021	750.	1.101	1000.	1.293	1250.	1.624	1500.	2.098	1750.	2.703	2000.	3.428	2250.	4.270	2500.	5.221
10,000					1500.	1.118	2000.	1.337	2500.	1.709	3000.	2.233	3500.	2.805	4000.	3.686	4500.	4.601	5000.	5.635
15,000	750.	1.0017	1500.	1.026	2250.	1.107	3000.	1.361	3750.	1.754	4500.	2.304	5250.	2.997	6000.	3.824	6750.	4.777	7500.	5.852
20,000	1000.	1.0018	2000.	1.028	3000.	1.134	4000.	1.379	5000.	1.789	6000.	2.359	7000.	3.075	8000.	3.927	9000.	4.910	10000.	6.018



Curve-Plate IV.

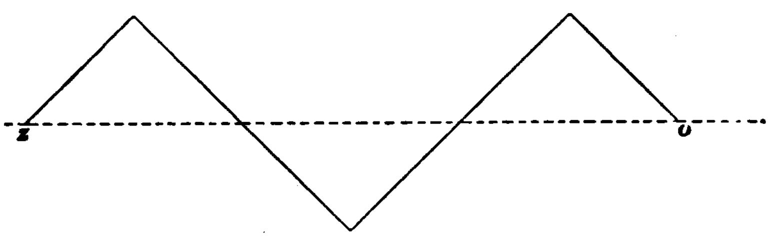


Fig. 7.

penetrate the soil. As a fact, the current cannot be confined to this superficial layer, and the real equivalent loop must be considerably wider than the hypothetical image requires. However, as the real limits can scarcely be assigned, the image impedance factors may be justly regarded as mimima, while the increase in the factor is usually very slow with moderate elevations even for large changes in the interaxial distance, so that the error is not so great as might be deemed on first apprehension. The minimum impedance factor for a wire say 20 feet above ground, is therefore, its loop impedance factor at 40 feet between axes. The curves indeed could not conveniently be extended, at their existing scale of construction, to include overhead wires, but the tables have probably sufficient range for practical purposes in this respect.

We may next examine how far these tabulated impedance factors are liable to be affected by waiving the restrictions hitherto maintained concerning the type of current waves, and the non-ferric nature of the inductances.

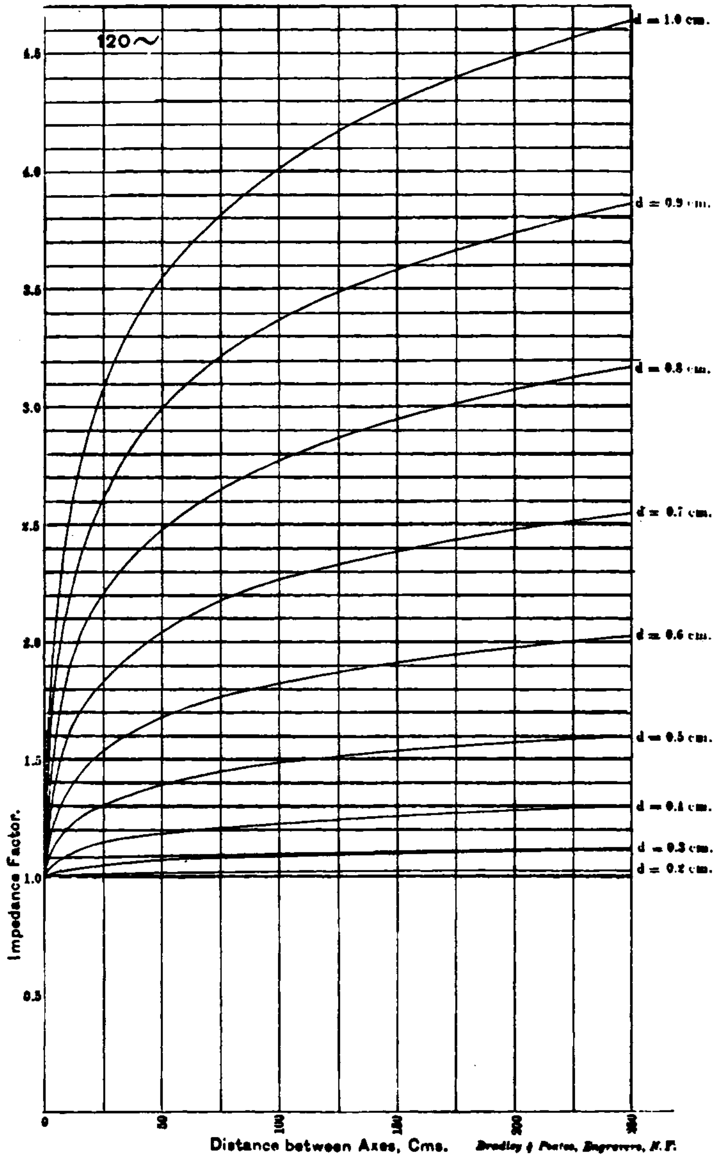
Impedance factors for conductors remote from iron, while independent of the current strength, depend immediately upon the current wave character. In practice, alternating current circuits contain motors or transformers, in which iron plays a prominent part; and owing to the hysteresis in this iron, the current waves deviate from simple sinusoids, particularly at light loads, even when the E.M.F. supplied by the generator is simply harmonic. Of all possible current wave types, however, the simply harmonic or sinusoidal has the lowest impedance factor, and the entries in the tables are consequently minimum values; but there appears to be no limit to the possible augmentation of these factors under appropriate conditions, as an example will indicate.

Let the current waves be of the saw-tooth type [Fig. 7], the

TABLE IV.

IMPEDANCE FACTORS FOR PAIRS OF PARALLEL COPPER WIRES CARRYING SIMPLE HARMONIC CURRENTS OF 80 ~ AT 15°C.

	Diam. 0.1 cm. Rad. 0.05 cm.		Diam. 0.2 cm. Rad. 0.1 cm.		Diam. 0.3 cm. Rad. 0.15 cm.		Diam. 0.4 cm. Rad. 0.2 cm.		Diam. 0.5 cm. Rad. 0.25 cm.		Diam. 0.6 cm. Rad. 0.3 cm.		Diam. 0.7 cm. Rad. 0.35 cm.		Diam. 0.8 cm. Rad. 0.4 cm.		Diam. 0.9 cm. Rad. 0.45 cm.		Diam. 1. cm. Rad. 0.5 cm.	
$\frac{D.}{r}$	*D.*	factor.	*D.*	factor.	*D.*	factor.	*D.*	factor.	*D.*	factor.	*D.*	factor.	*D.*	factor.	*D.*	factor.	*D.*	factor.	*D.*	factor.
2			0.2	1.0002	0.3	1.0008	0.4	1.0028	0.5	1.0061	0.6	1.0126	0.7	1.023	0.8	1.039	0.9	1.062	1.	1.093
3							0.6	1.0050	0.75	1.0124	0.9	1.026	1.05	1.047	1.2	1.078	1.35	1.123	1.5	1.182
4					0.6	1.0024	0.8	1.0075	1.	1.018	1.2	1.037	1.4	1.068	1.6	1.114	1.8	1.177	2.	1.259
5			0.5	1.0006			1.	1.0097	1.25	1.024	1.5	1.048	1.75	1.087	2.	1.144	2.25	1.223	2.5	1.325
6					0.9	1.0037	1.2	1.0117	1.5	1.027	1.8	1.058	2.1	1.104	2.4	1.172	2.7	1.264	3.	1.383
8							1.6	1.015	2.	1.037	2.4	1.074	2.8	1.134	3.2	1.219	3.6	1.334	4.	1.479
10			1.	1.0011	1.5	1.0057	2.	1.018	2.5	1.044	3.	1.089	3.5	1.158	4.	1.259	4.5	1.391	5.	1.557
12							2.4	1.021	3.	1.050	3.6	1.101	4.2	1.180	4.8	1.292	5.4	1.440	6.	1.623
16							3.2	1.025	4.	1.061	4.8	1.122	5.6	1.216	6.4	1.348	7.2	1.520	8.	1.731
20			2.	1.0018	3.	1.008	4.	1.029	5.	1.070	6.	1.140	7.	1.246	8.	1.394	9.	1.585	10.	1.817
25							5.	1.033	6.25	1.079	7.5	1.158	8.75	1.278	10.	1.441	11.25	1.651	12.	1.905
30					4.5	1.012	6.	1.037	7.5	1.087	9.	1.174	10.5	1.304	12.	1.481	13.5	1.706	15.	1.979
40			4.	1.0027			8.	1.042	10.	1.101	12.	1.20	14.	1.347	16.	1.546	18.	1.796	20.	2.096
60					9.	1.017	12.	1.051	15.	1.121	18.	1.239	21.	1.411	24.	1.640	27.	1.925	30.	2.264
80							16.	1.058	20.	1.137	24.	1.268	28.	1.458	32.	1.709	36.	2.0915	40.	2.385
100			10.	1.0041	15.	1.021	20.	1.064	25.	1.150	30.	1.291	35.	1.496	40.	1.764	45.	2.093	50.	2.481
150	7.5	1.0003					30.	1.075	37.5	1.174	45.	1.336	52.5	1.566	60.	1.865	67.5	2.229	75.	2.655
200					30.	1.027	40.	1.083	50.	1.192	60.	1.368	70.	1.617	80.	1.938	90.	2.354	100.	2.780
300			30.	1.0062			60.	1.095	75.	1.218	90.	1.416	105.	1.692	120.	2.043	135.	2.467	150.	2.959
400	20.	1.0004			60.	1.034	80.	1.104	100.	1.238	120.	1.451	140.	1.745	160.	2.119	180.	2.567	200.	3.085
500							100.	1.111	125.	1.253	150.	1.478	175.	1.787	200.	2.178	225.	2.645	250.	3.184
600			60.	1.0077	90.	1.039	120.	1.117	150.	1.266	180.	1.501	210.	1.822	240.	2.226	270.	2.708	300.	3.265
700	35.	1.0005					140.	1.122	175.	1.277	210.	1.520	245.	1.851	280.	2.268	315.	2.763	350.	3.333
800					120.	1.042	160.	1.127	200.	1.287	240.	1.537	280.	1.877	320.	2.304	360.	2.811	400.	3.394
900			90.	1.0087			180.	1.131	225.	1.296	270.	1.552	315.	1.900	360.	2.336	405.	2.853	450.	3.447
1,000	50.	1.0006			150.	1.044	200.	1.134	250.	1.304	300.	1.566	350.	1.920	400.	2.364	450.	2.890	500.	3.493
1,500							300.	1.149	375.	1.335	450.	1.619	525.	2.000	600.	2.474	675.	3.034	750.	3.676
2,000			200.	1.011	300.	1.053	400.	1.160	500.	1.357	600.	1.657	700.	2.058	800.	2.554	900.	3.138	1000.	3.805
3,000	150.	1.0007	300.	1.012	450.	1.059	600.	1.175	750.	1.390	900.	1.712	1050.	2.140	1200.	2.666	1350.	3.284	1500.	3.989
4,000			400.	1.013	600.	1.063	800.	1.187	1000.	1.413	1200.	1.751	1400.	2.198	1600.	2.745	1800.	3.387	2000.	4.118
5,000	250.	1.0008	500.	1.013	750.	1.066	1000.	1.196	1250.	1.431	1500.	1.783	1750.	2.243	2000.	2.808	2250.	3.467	2500.	4.219
10,000			1000.	1.016	1500.	1.077	2000.	1.226	2500.	1.493	3000.	1.885	3500.	2.393	4000.	3.010	4500.	3.730	5000.	4.547
15,000			1500.	1.017	2250.	1.083	3000.	1.243	3750.	1.526	4500.	1.938	5250.	2.471	6000.	3.117	6750.	3.868	7500.	4.720
20,000	1000.	1.0012	2000.	1.018	3000.	1.088	4000.	1.256	5000.	1.552	6000.	1.980	7000.	2.532	8000.	3.198	9000.	3.973	10000.	4.852



Curve-Plate V.

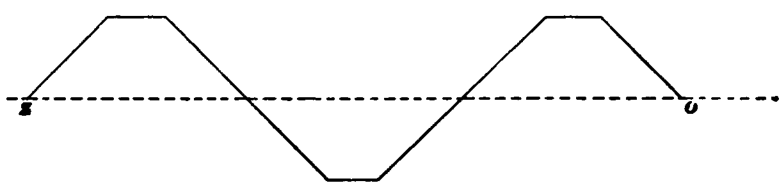


Fig. 8.

rising and falling lines intersecting the zero line ZO at equal angles but in opposite directions. Here the time occupied in ascent from, or descent to, the zero line, is just one quarter period. The impedance factor for such currents is

$$\sqrt{1+\frac{48n^2l^2}{r^2}}$$

where n is the frequency and l/r the ratio of inductance to resistance in any length of conductor, or in other words its time constant. The tabulated factors for simple harmonic currents we know to be

$$\sqrt{1+\frac{p^2l^2}{r^2}},$$

$$\text{or } \sqrt{1+39.48\frac{n^2l^2}{r^2}},$$

so that the saw-tooth factors are the greater. If we file down the points of the teeth as shown in Fig. 8, the current will rise from zero to a maximum, along the straight line OA in less than $\frac{1}{4}$ period, and the factor diminishes, until this time of ascent is reduced to $\frac{3}{16}$ of a period [indicated in Fig. 8], when the factor has its minimum value for this character of wave, viz.:

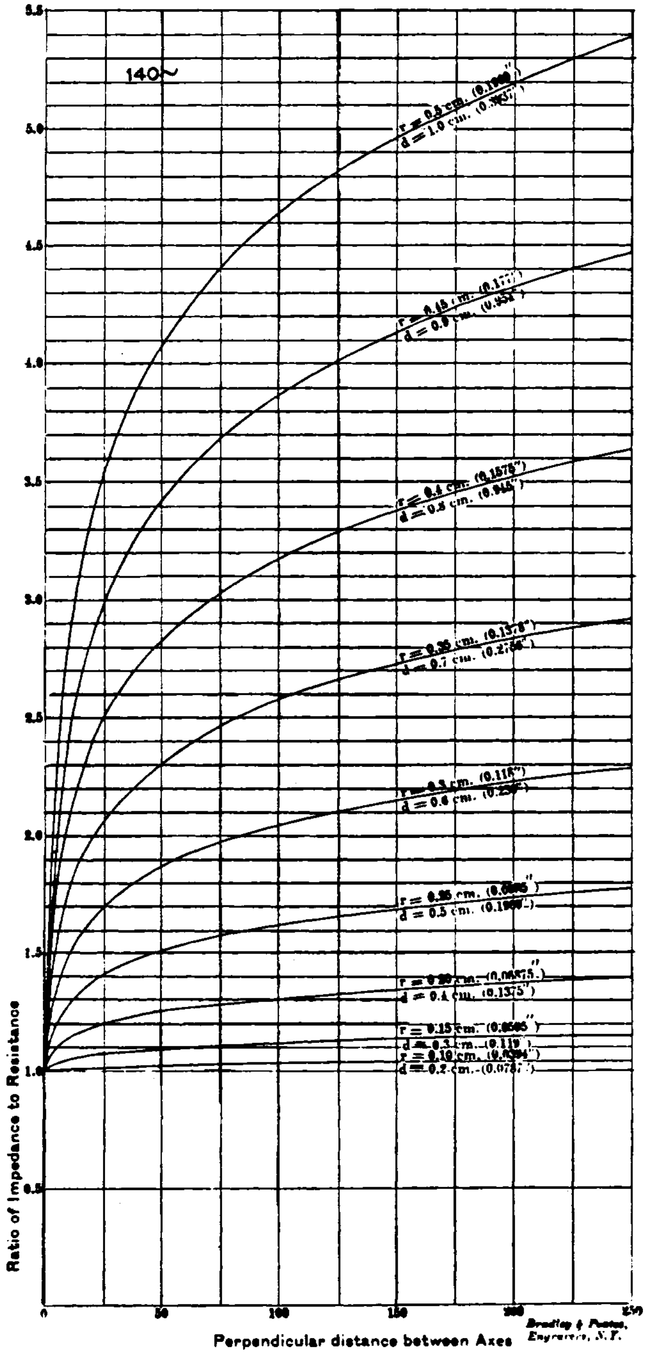
$$\sqrt{1+43.2\frac{l^2n^2}{r^2}}.$$

After this, as we shorten the period of ascent, the factor rises.

TABLE V.

IMPEDANCE FACTORS FOR PAIRS OF PARALLEL COPPER WIRES CARRYING SIMPLE HARMONIC CURRENTS OF 60 ~ AT 15°C.

	Diam. 0.1 cm. Rad. 0.05 cm.		Diam. 0.2 cm. Rad. 0.1 cm.		Diam. 0.3 cm. Rad. 0.15 cm.		Diam. 0.4 cm. Rad. 0.2 cm.		Diam. 0.5 cm. Rad. 0.25 cm.		Diam. 0.6 cm. Rad. 0.3 cm.		Diam. 0.7 cm. Rad. 0.35 cm.		Diam. 0.8 cm. Rad. 0.4 cm.		Diam. 0.9 cm. Rad. 0.45 cm.		Diam. 1. cm. Rad. 0.5 cm.	
$\frac{D.}{r}$	*D.*	factor.	*D.*	factor.	*D.*	factor.	*D.*	factor.	*D.*	factor.	*D.*	factor.	*D.*	factor.	*D.*	factor.	*D.*	factor.	*D.*	factor.
2			0.2	1.0001	0.3	1.0004	0.4	1.0014	0.5	1.0034	0.6	1.007	0.7	1.0313	0.8	1.0224	0.9	1.036	1.	1.046
3									0.75	1.007	0.9	1.015	1.05	1.0265	1.2	1.045	1.35	1.071	1.5	1.106
4							0.8	1.0042	1.	1.010	1.2	1.021	1.4	1.039	1.6	1.066	1.8	1.103	2.	1.153
5					0.75	1.0017			1.25	1.013	1.5	1.028	1.75	1.050	2.	1.084	2.25	1.131	2.5	1.194
6							1.2	1.0066	1.5	1.016	1.8	1.033	2.1	1.064	2.4	1.100	2.7	1.156	3.	1.230
8									2.	1.021	2.4	1.043	2.8	1.077	3.2	1.129	3.6	1.199	4.	1.291
10			1.0	1.0006	1.5	1.0033	2.	1.0103	2.5	1.025	3.	1.051	3.5	1.092	4.	1.153	4.5	1.235	5.	1.342
12									3.	1.029	3.6	1.058	4.2	1.105	4.8	1.173	5.4	1.266	6.	1.386
16							3.2	1.0144	4.	1.035	4.8	1.070	5.6	1.127	6.4	1.209	7.2	1.318	8.	1.458
20					3.	1.0053			5.	1.040	6.	1.081	7.	1.145	8.	1.238	9.	1.360	10.	1.515
25							5.	1.019	6.25	1.045	7.5	1.092	8.75	1.164	10.	1.268	11.25	1.404	12.	1.575
30									7.5	1.050	9.	1.101	10.5	1.181	12.	1.293	13.5	1.441	15.	1.625
40			4.	1.0015	6.	1.0077	8.	1.024	10.	1.059	12.	1.117	14.	1.207	16.	1.335	18.	1.501	20.	1.705
60									15.	1.070	18.	1.140	21.	1.248	24.	1.397	27.	1.588	30.	1.823
80							16.	1.034	20.	1.079	24.	1.158	28.	1.278	32.	1.443	36.	1.653	40.	1.908
100					15.	1.012			25.	1.087	30.	1.173	35.	1.302	40.	1.479	45.	1.704	50.	1.975
150							30.	1.043	37.5	1.101	45.	1.201	52.5	1.348	60.	1.547	67.5	1.798	75.	2.099
200									50.	1.112	60.	1.221	70.	1.381	80.	1.597	90.	1.866	100.	2.187
300			30.	1.004	45.	1.018	60.	1.057	75.	1.128	90.	1.251	105.	1.431	120.	1.669	135.5	1.965	150.	2.315
400									100.	1.140	120.	1.273	140.	1.466	160.	1.721	180.	2.036	200.	2.406
500							100.	1.064	125.	1.149	150.	1.291	175.	1.495	200.	1.763	225.	2.091	250.	2.478
600					90.	1.022			150.	1.157	180.	1.305	210.	1.518	240.	1.796	270.	2.136	300.	2.537
700							140.	1.070	175.	1.164	210.	1.318	245.	1.538	280.	1.825	315.	2.175	350.	2.586
800									200.	1.170	240.	1.329	280.	1.556	320.	1.850	360.	2.210	400.	2.630
900			90.	1.005	135.	1.025	180.	1.076	225.	1.176	270.	1.339	315.	1.571	360.	1.873	405.	2.239	450.	2.668
1,000							200.	1.078	250.	1.181	300.	1.348	350.	1.585	400.	1.893	450.	2.265	500.	2.702
1,500							300.	1.086	375.	1.199	450.	1.383	525.	1.640	600.	1.971	675.	2.369	750.	2.835
2,000					300.	1.031	400.	1.093	500.	1.214	600.	1.408	700.	1.679	800.	2.027	900.	2.444	1000.	2.930
3,000			300.	1.007			600.	1.102	750.	1.234	900.	1.444	1050.	1.736	1200.	2.106	1350.	2.549	1500.	3.064
4,000							800.	1.109	1000.	1.249	1200.	1.471	1400.	1.776	1600.	2.163	1800.	2.625	2000.	3.159
5,000	250.	1.0005			750.	1.037	1000.	1.114	1250.	1.261	1500.	1.492	1750.	1.808	2000.	2.207	2250.	2.683	2500.	3.233
10,000			1000.	1.009			2000.	1.133	2500.	1.300	3000.	1.560	3500.	1.912	4000.	2.353	4500.	2.874	5000.	3.476
15,000							3000.	1.143	3750.	1.322	4500.	1.597	5250.	1.968	6000.	2.430	6750.	2.975	7500.	3.601
20,000	1000.	1.0006	2000.	1.019	3000.	1.050	4000.	1.151	5000.	1.339	6000.	1.626	7000.	2.011	8000.	2.489	9000.	3.052	10000.	3.698



Curve-Plate VI.

Neither wave amplitude nor strength of current can of course affect this consideration.

$$\text{When asc. and des. each occ. } \tfrac{1}{8} \text{ period, factor} = \sqrt{1 + 48\frac{l^2n^2}{r^2}}\,;$$

$$\text{'' '' '' '' '' '' } \tfrac{1}{10} \text{ '' '' } = \sqrt{1 + 54.5\frac{l^2n^2}{r^2}}\,;$$

$$\text{'' '' '' '' '' '' } \tfrac{1}{12} \text{ '' '' } = \sqrt{1 + 62\frac{l^2n^2}{r^2}}\,;$$

$$\text{'' '' '' '' '' '' } \tfrac{1}{100} \text{ '' '' } = \sqrt{1 + 411\frac{l^2n^2}{r^2}}\,,$$

and finally, when the ascending time is reduced to zero, so that the rise is absolutely perpendicular, as shown in Fig. 9, the factor becomes indefinitely great, and the drop on such a wire would no longer be finite with a finite current, which is equivalent to the statement that it is practically impossible to create currents of this strictly rectangular type.

Since theory assigns no limits, we are compelled to accept the terms that existing practice may impose. Fig. 10 shows a wave form taken from Professor Ryan's paper on Transformers[4] read before this Institute in January, 1890. It was the wave of primary current supplied to an idle transformer by an alternator whose E.M.F. was nearly sinusoidal. This wave has been analyzed into its harmonic components, given in Appendix II. We know from Fourier's beautiful theorem, that any periodic, non-re-entrant wave, however complex in outline, can be compounded by, or analyzed into, simple sine waves whose frequencies are all some multiple (integral and not fractional) of the resultant or complex wave frequency. It is evident that any wave, however irregular in form, can be constructed out of a sine wave of the same length, with sine ripples of sufficient number and diminutiveness added to or subtracted from this fundamental sinusoid at suitable points, but it is wonderful that the process can always be completed with sine ripples, each belonging to an endless succession of its own type, and each ripple-chain having a wave length some submultiple of the fundamental. The impedance factor for any complex harmonic current wave is always determinable from the formula of Appendix II, when its dissection into component ripples, that is to say its harmonic analysis, has been determined. In this instance the harmonics and fundamental wave that approximately compound into the current wave considered, are indicated by the dotted lines of Fig. 10, and the impedance factor is

$$\sqrt{1 + 1.605\frac{p^2l^2}{r^2}} \quad \text{or} \quad \sqrt{1 + 63.35\frac{n^2l^2}{r^2}}$$

so that a pair of No. 2 A.W.G. wires interaxially distant 30 in. would offer an impedance factor to this current wave at 140 ~ of 2.708, an increase of nearly 22 per cent. above the tabular value for simple sinusoids. It is evident that the drop in voltage over such wires, or over any suitable coil of known resistance and non-ferric inductance, would furnish evidence as to whether the current wave type differs sensibly from the simply harmonic. The sensitiveness of the test increases with the frequency, and with the time constant of the coil. Thus the wave in Fig. 10 which adds 22 per cent. to a normal factor of 2.22, adds only 8 per cent. to a normal factor of 1.18 at 140 ~ , such as would be furnished by two No. 8 A.W.G. wires 66 in. apart.

Professor Ryan's paper shows, however, that this distortion in the current wave through the primary circuit of the transformer, gradually disappeared as the load was added to the secondary circuit, and in fact at full load, the current wave was nearly sinusoidal, with an impedance factor close to the normal or tabular value. This condition is the more fortunate, since in arranging supply circuits, it is only the full load drop that is usually of consequence, and for this it would seem that the tabular impedance factors are closely applicable, provided that the wave of E.M.F. generated by the alternator can be regarded as sinusoidal.

The impedance factor for an overhead line wire of copper is the same, whether the current it carries is returned through one or through many similar wires, provided that in the latter case the return wires are all equally distant from it. In other

[4] TRANSACTIONS, vol. vii, p. 1.

TABLE VI.

IMPEDANCE FACTORS FOR PAIRS OF PARALLEL COPPER WIRE CARRYING SIMPLE HARMONIC CURRENTS OF 40 ~ AT 15°C.

	Diam. 0.1 cm. Rad. 0.05 cm. D.		Diam. 0.2 cm. Rad. 0.1 cm. D.		Diam. 0.3 cm. Rad. 0.15 cm. D.		Diam. 0.4 cm. Rad. 0.2 cm. D.		Diam. 0.5 cm. Rad. 0.25 cm. D.		Diam. 0.6 cm. Rad. 0.3 cm. D.		Diam. 0.7 cm. Rad. 0.35 cm. D.		Diam. 0.8 cm. Rad. 0.4 cm. D.		Diam. 0.9 cm. Rad. 0.45 cm. D.		Diam. 1. cm. Rad. 0.5 cm. D.	
$\frac{D.}{r}$	cms.	factor.	cms.	factor.	cms.	factor.	cms.	factor.	cms.	factor.	cms.	factor.	cms.	factor.	cms.	factor.	cms.	factor.	cms.	factor.
2			0.2	1.00004	0.3	1.0002	0.4	1.0006	0.5	1.0015	0.6	1.0032	0.7	1.0058	0.8	1.010	0.9	1.016	1.	1.024
3									0.75	1.0031	0.9	1.0064	1.05	1.012	1.2	1.020	1.35	1.032	1.5	1.048
4							0.8	1.0019	1.	1.0047	1.2	1.0095	1.4	1.018	1.6	1.030	1.8	1.047	2.0	1.071
5					.075	1.0008			1.25	1.0059	1.5	1.012	1.75	1.023	2.	1.038	2.25	1.060	2.5	1.090
6							1.2	1.0029	1.5	1.0071	1.8	1.015	2.1	1.027	2.4	1.045	2.7	1.072	3.	1.108
8									2.	1.0092	2.4	1.019	2.8	1.035	3.2	1.059	3.6	1.093	4.	1.138
10			1.	1.0003	1.5	1.0014	2.	1.0046	2.5	1.011	3.	1.023	3.5	1.042	4.	1.070	4.5	1.111	5.	1.165
12									3.	1.013	3.6	1.026	4.2	1.048	4.8	1.081	5.4	1.126	6.	1.187
16							3.2	1.0064	4.	1.016	4.8	1.032	5.6	1.058	6.4	1.098	7.2	1.152	8.	1.225
20					3.	1.0023			5.	1.018	6.	1.037	7.	1.067	8.	1.111	9.	1.174	10.	1.256
25							5.	1.0084	6.25	1.021	7.5	1.042	8.75	1.076	10.	1.127	11.25	1.197	12.5	1.288
30									7.5	1.022	9.	1.046	10.5	1.084	12.	1.140	13.5	1.216	15.	1.315
40			4.	1.0007	6.	1.0034	8.	1.011	10.	1.026	12.	1.053	14.	1.097	16.	1.161	18.	1.248	20.	1.360
60									15.	1.032	18.	1.065	21.	1.117	24.	1.193	27.	1.295	30.	1.426
80							10.	1.015	20.	1.036	24.	1.073	28.	1.132	32.	1.217	36.	1.330	40.	1.474
100					15.	1.005			25.	1.039	30.	1.080	35.	1.144	40.	1.236	45.	1.358	50.	1.513
150							30.	1.019	37.5	1.046	45.	1.094	52.5	1.168	60.	1.273	7.65	1.412	75.	1.585
200									50.	1.051	60.	1.104	70.	1.185	80.	1.300	90.	1.450	100.	1.638
300			30.	1.0016	45.	1.008	60.	1.025	75.	1.059	90.	1.119	105.	1.211	120.	1.339	135.	1.507	150.	1.714
400									100.	1.064	120.	1.130	140.	1.230	160.	1.368	180.	1.548	200.	1.769
500							100.	1.029	125.	1.069	150.	1.138	175.	1.244	200.	1.391	225.	1.581	250.	1.812
600					90.	1.010			150.	1.073	180.	1.146	210.	1.257	240.	1.411	270.	1.608	300.	1.848
700							140.	1.032	175.	1.076	210.	1.152	245.	1.268	280.	1.427	315.	1.630	350.	1.878
800									180.	1.079	220.	1.158	260.	1.277	320.	1.441	360.	1.651	400.	1.905
900			90.	1.0022	135.	1.011	180.	1.034	225.	1.082	270.	1.163	305.	1.286	350.	1.454	395.	1.669	450.	1.929
1,000									250.	1.084	300.	1.167	350.	1.293	400.	1.465	450.	1.685	500.	1.949
1,500							275.	1.039	350.	1.093	425.	1.185	500.	1.323	575.	1.510	650.	1.747	725.	2.031
2,000					300.	1.014	400.	1.042	500.	1.100	600.	1.198	700.	1.345	800.	1.543	900.	1.792	1000.	2.090
3,000			300.	1.003	450.	1.015	600.	1.047	750.	1.110	900.	1.218	1050.	1.360	1200.	1.589	1350.	1.856	1500.	2.174
4,000					600.	1.016	800.	1.050	1000.	1.118	1200.	1.232	1400.	1.399	1600.	1.623	1800.	1.902	2000.	2.234
5,000	250.	1.0002			750.	1.017	1000.	1.052	1250.	1.124	1500.	1.243	1750.	1.417	2000.	1.649	2250.	1.938	2500.	2.280
10,000					1500.	1.019	2000.	1.061	2500.	1.143	3000.	1.280	3500.	1.477	4000.	1.736	4500.	2.058	5000.	2.433
15,000			1500.	1.0042	2250.	1.021	3000.	1.066	3750.	1.154	4500.	1.299	5250.	1.509	6000.	1.783	6750.	2.119	7500.	2.514
20,000	1000.	1.0003	2000.	1.0045	3000.	1.022	4000.	1.070	5000.	1.163	6000.	1.315	7000.	1.534	8000.	1.819	9000.	2.167	10000.	2.576

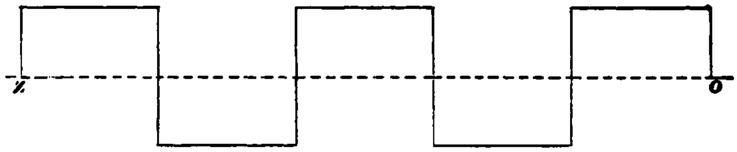


FIG. 9.

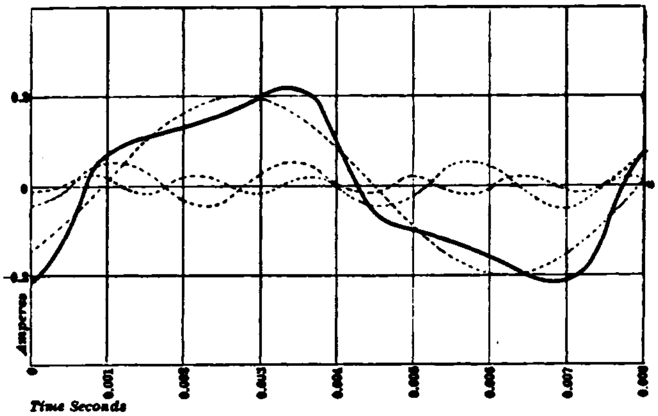


FIG. 10.

words, if with any single outgoing wire as axis, we trace a cylinder in space, the impedance factor of that wire will be the same whether its return circuit be completed through one or many wires of its own size, provided that all such return wires have their axes located on the cylindrical surface, and independent of the proportions of return current that these may individually carry. In the extreme case the return wires will all be in contact side by side, and form a continuous shell or cylinder round the outgoing conductor at the centre, whose inductance and consequent impedance remains the same as when all the returns are removed save one. It follows, therefore, that the inductance of an insulated copper wire of radius r, concentrically surrounded, (or even, it happens, eccentrically), by a very thin conducting cylindrical sheath of radius R has the impedance factor of the same wire when forming one side of a looped pair at interaxial distance R. The inductances and impedance factors of the return conductors in these cases when separated or dispersed, are always much reduced, and in this last example, the inductance of the exterior conducting shell diminishes indefinitely with the thinness of the cylinder, its impedance factor tending coincidently to the limit unity.

For the same reasons the tables and curves also apply to three parallel copper wires carrying triphase sine currents of equal effective strength, provided that the three wires are equidistant. The distance between any one of the three pairs will then correspond to the ordinary loop distance.

All the results thus far considered and tabulated have been based upon the assumption that the current density in the conductors remains uniform. It is well known that with alternating currents, as the size of the wire and the frequency increase, the current waves cease to penetrate with full amplitude to the axis, but tend to desert the interior for the more superficial layers, thus increasing the apparent resistance of the wire while diminishing its inductance. The basis of this phenomenon of incomplete penetration involves the theory of the transmission of electric energy through space, as well as the relative share of the conductor and its environing dielectric in this activity; but without exploring fundamental causes, the facts are readily explained by causes more proximate and familiar. If we cause magnetic flux to oscillate round a straight conductor, an induced alternating E.M.F. will be set up along that conductor, as in any alternating transformer. But an alternating current sent through the wire would itself set up an oscillating field of this nature about the wire's axis, and the current would be opposed by the induced E.M.F. it thus establishes in the substance of the conductor. This E.M.F. would be greatest at the axis, for at the surface only the flux sweeping round the wire externally, generates voltage, while as we descend towards the centre, the flux within the wire adds to the effect, particularly if the wire is of iron or nickel. At the axis, all the flux in and beyond the wire unites to generate E.M.F. in oscillation. Under the resultant influence of this, and the impressed voltage at the terminals of the wire, the current will be strongest where the induced counter E.M.F. is weakest, namely at the exterior surface, and its distribution must be such that the "drop" will just sustain a balance between this induced and the impressed E.M.F's. at every point and instant.

This extra virtual resistance by imperfect penetration is generally negligible in ordinary sizes of copper wires at the frequencies employed in practice. Thus with No. 000 A.W.G., the largest wire in the tables, this extra resistance is 1.6 per cent. at 140 ~ , 1.2 per cent. at 120 ~ , and 0.8 per cent at 100 ~ ; while the impedance factors, combining this with the inductance are affected in a still lesser degree. Again for No. 00 A.W.G., the extra resistance is 1.0 per cent. at 140 ~ , 0.75 per cent. at 120 ~ , and 0.5 per cent. at 100 ~ , so that for nearly all purposes we may neglect this influence and accept the results embodied in the curves and tables, wherein perfect penetration is assumed.

But in iron wires this virtual extra resistance is known to attain large proportions within practical limits of size and frequency. Figs. 11 and 12 represent a series of observations upon iron wires carrying current waves approximately sinusoidal,[5] at 140 ~ . These iron wires were stretched in a long loop upon the testing room floor, the interaxial loop distance being 10 cms. The impedance factors of the wires in

[5]The analysis of the wave of E.M.F., generated by the alternator in these and other experiments described, was made by a wiping contact and static voltmeter, over 100 subdivisions of the cycle, and the plotted observations nowhere differed from a sinusoid by more than 3%.

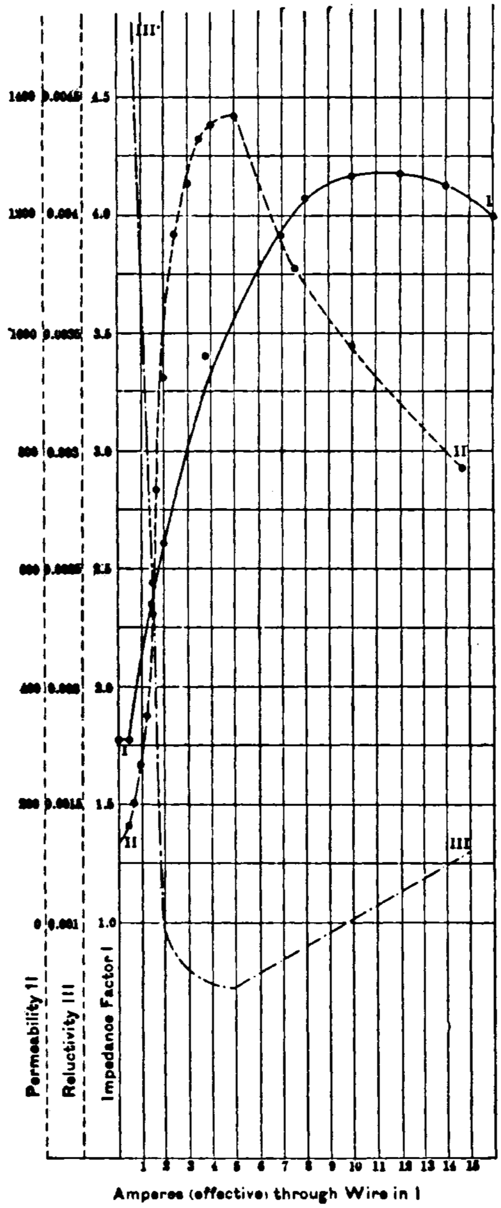


Fig. 11.

Curve I., Observed Impedance Factors; II., Observed Permeability; III., Observed Reluctivity.
Particulars: No. 5, B. W. G. soft iron wire, diameter 0.566 cm., stretched in one straight loop 2785 cms. long, with 10 cm. interaxial distance.
Resistance of iron wire 47.47 microhms per linear cm. @ 15° C.
Resistivity 11.84 microhms @ 15° C.

copper under these circumstances would have been 1.42 for No. 5 B.W.G. and 1.1 for No. 9 B.W.G. The impedance factors commenced at values not far above these, but increased with the current to a maximum about ten amperes for the larger wire, and 7.5 amperes with the smaller, while beyond those currents the factors diminished. These impedances were compared with that of a stout german silver wire in the form of a long loop, and with the aid of a differential dynamometer,[6] each of the two separate dynamometers in the complete differential instrument being connected by "pressure wires" with either the standard loop of german silver or the measured loop of iron wire, the connections being occasionally interchanged as a check. As the current in the loops varied, the electrodes on the iron wire loop had to be shifted, in order to restore the differential balance. In this way reliable results could be obtained with 55 metres of stout iron wire, and so long as the temperature of the iron is not sensibly elevated, the results are definite, and retain, within the limits of observation, the same values in ascending or descending series of current strengths. By welding short lengths of the two samples of wire into circular loops, and wrapping these with wire, their permeability and reluctivity were determined by ballistic galvanometer according to the curves shown.

A similar impedance test of a loop of No. 14 A.W.G. iron wire (diameter 0.16 cm.) showed that the impedance did not sensibly alter between current strengths of 0.2 and 2.0 amperes, the latter being the limit imposed by elevation of temperature. In this case with an interaxial distance of 19 cms. and 140 ~ , the observed factor was 1.03.

The influence of imperfect penetration on the impedance of iron wires has been calculated by Rayleigh and Heaviside on

[6] This instrument has been lately fully described in the *Electrical Engineer*.

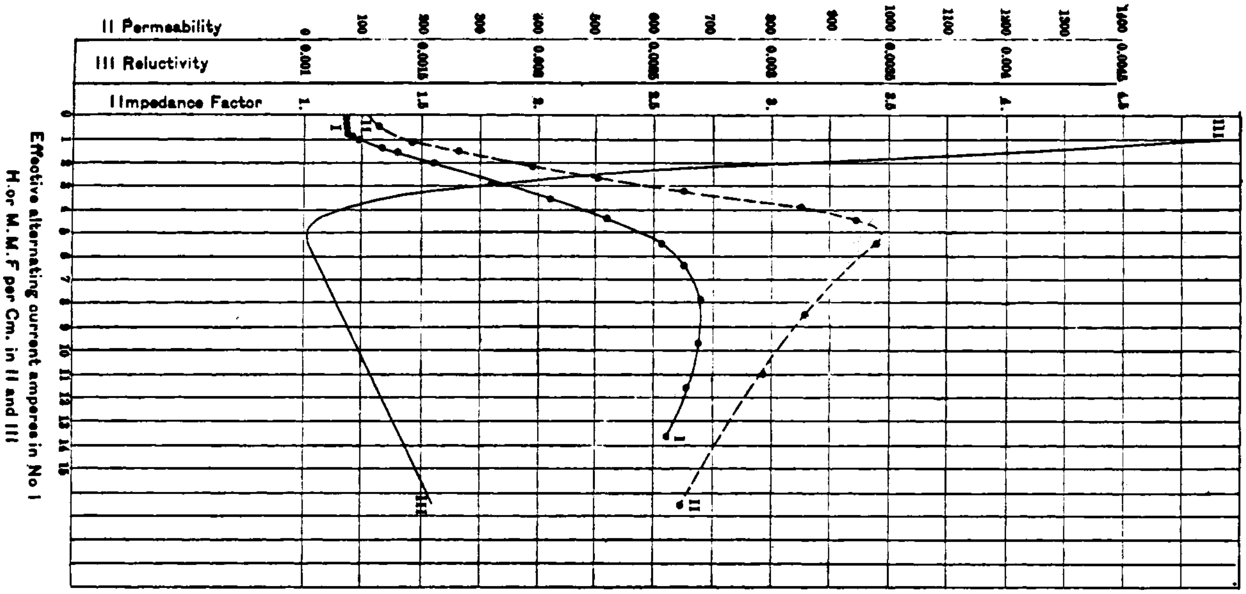


FIG. 13.

Curves of impedance and permeability observed in a galvanized iron wire No. 9, B. W. G. Diameter 0.148" = 0.375 cm. drawn into one straight loop 2690 cms. long and 10 cms. interaxial distance. Specific gravity of wire, 7.76. Impedance of loop compared with a loop of german silver wire of diam. 0.28 cm. 1,000 cms. long and 7 cms interaxial distance by differential dynamometer. E. M. F. of generator very closely sinusoidal. Frequency 140 ∼. Temp. of wire under test 10° C. Resistance of iron wire per linear cm. 208 microhms. Resistivity 12.09 microhms at 10° C.

Curve I., Impedance factor as function of current; II., Permeability as function of *H*; III., Reluctivity as function of *H*.

the basis of a uniform permeability through the wire. In reality the results are still more complex, owing to the variation of the permeability at different distances from the centre.

A number of measurements upon impedance in iron wires of the quality in ordinary use for telegraphic and telephonic use, seem to show that with comparatively feeble currents, the mean permeability of the iron is about 150. This means that the inductance of an iron wire is 12 millihenrys per mile greater than the inductance would be if the wire were of copper. On page 10 of Vol. VIII. of the TRANSACTIONS of this Institute, a table is given of the inductance of suspended single wires of copper with ground return. The average value is about 3 millihenrys per mile. If these are increased by 12 millihenrys, the entries should correspond to the average inductances of iron wires under similar conditions (i.e., about 15 millihenrys per mile), except that for wires of more than 1 mm. radius or 2 mms. diameter—say No. 12 A.W.G.—the influence of imperfect penetration commences to be serious, and impedances calculated from these inductances are only reliable for iron wires of No. 12 A.W.G. or smaller.

Continuing, Mr. Kennelly added orally:

In conclusion, to sum up briefly in inverted order:—First the inductance of a telephone iron wire is about five times as great as the inductance of that wire if it were copper. From that inductance, the impedance of the wire for various telephonic frequencies can be determined.

Secondly, the impedance of a larger iron wire, wherein imperfect current penetration plays a considerable part, is such that it increases to a maximum with a certain current, and then diminishes as that current is increased.

Thirdly, the impedance of copper wires such as are used in the distribution of electric light and power on alternating current systems can be determined, if the current waves are known to be practically sinusoidal, when the distance of these wires apart, and their sizes are known. Take the drop that would be established by the current if continuous, and multiply by the factor as shown in these tables and curves.

And, fourthly, and with most emphasis of all, the paper has been written with the endeavor to point out that the difficulties which at present enshroud the use and the working theory of alternating currents are largely fictitious. An alternating current circuit is certainly a very complex thing; but so is a dynamo, and we can predetermine dynamos with a degree of accuracy sufficient for all practical purposes. We know when a machine is designed, within reasonable limits, what the voltage will be at its terminals when driven at a certain speed with a certain excitation and load, although the exact calculation would transcend all the capabilities we possess at this day. Similarly, working theories which are sufficiently good for most practical purposes run through all the kindred sciences, whereas the exact theories are beyond present human capabilities; and although alternating currents are so difficult when studied accurately and absolutely, the working theory of alternating currents can be made as simple as the working theory of continuous currents. I am firmly impressed with the belief —a belief which I trust I may be able to communicate—that the most convincing way of proving that this difficulty which has hitherto surrounded the alternating current and its distribution, can be eliminated and removed is, by the development of the notion of impedance.

APPENDIX I.

Determination of the inductance of a loop formed by two indefinitely long parallel copper wires each of radius r cms. fixed at an interaxial distance d in a medium of inductivity μ. Let the substance of the wires have a resistivity ρ and inductivity μ_0. Let the two wires be denoted by the symbols A and D.

The inductance L (cms.) of a given length l (cms.) in either of these wires, say A, will be the total flux through the loop linked with unit current ($\gamma = 1$) flowing through that length.

This flux is partly external to the wire A and partly within its substance.

Considering first the external flux, the intensity B at any point

$$P \text{ is } B = \frac{2\gamma\mu}{\rho}$$

due to uniform current of γ units in A where ρ is the radial distance OP from the axis O of A. In a rectangle of length l and breadth $d\rho$ drawn in the plane through the wire axis, the total flux will be

$$F = Bld\rho = 2l\gamma\mu\frac{d\rho}{\rho}.$$

The total flux through the loop from the surface of A to the centre of D, so far as depends on A's current is the integral of this expression between the limits $\rho = r$ and $\rho = d$ or

$$F = 2l\gamma\mu \log \varepsilon \frac{d}{r}$$

and all of this flux encircles and is linked with the current γ making the current flux

$$2\mu l\gamma^2 \log \varepsilon \frac{d}{r}.$$

Within the substance of A, and at radius ρ, the intensity will be

$$B = \mu_0 \gamma \frac{\pi\rho^2}{\pi r^2} \cdot \frac{2}{\rho} = \frac{2\gamma\rho}{r^2}\mu_0.$$

In a similar rectangle of length l and breadth $d\rho$ the penetrating flux is

$$F = \frac{2l\gamma}{r^2}\mu_0 \rho d\rho$$

and this flux is not linked with all the current γ, but with that portion only which is within a cylinder of radius ρ namely with

$$\gamma\frac{\pi\rho^2}{\pi\gamma^2} = \gamma\frac{\rho^2}{r^2}$$

so that the current flux linkage is

$$\frac{2l\gamma^2}{r^4}\mu_0 \rho^3 d\rho$$

for the elementary rectangle. The total flux linkage within the wire will be the integral of this expression from

$$\rho = o \text{ to } \rho = r \text{ or } \frac{l\gamma^2}{2}\mu_0.$$

The total current flux linkage for the wire A is

$$\gamma^2 l\left(\frac{\mu_0}{2} + 2\mu \log \varepsilon \frac{d}{r}\right).$$

This is the inductance L of the length of l when $\gamma = 1$ so that

$$L = l\left(\frac{\mu_0}{2} + 2\mu \log \varepsilon \frac{d}{r}\right)$$

and the inductance per cm.

$$\lambda = \frac{L}{l} = \frac{\mu_0}{2} + 2\mu \log \varepsilon \frac{d}{r}.$$

By symmetry it will be the same for either A or D.

For wires in air μ is practically unity

$$\text{and } \lambda = \frac{\mu_0}{2} + 2 \log \varepsilon \frac{d}{r}.$$

For iron wire in air μ_0 is practically about 150 for currents of a few milliamperes from a number of measurements, so that

$$\lambda_{Fe} = \left(75 + 2 \log \varepsilon \frac{d}{r}\right) \text{ centimetres.}$$

For copper wire $\mu_0 = 1$ practically, and

$$\lambda_{Cu} = \left(\frac{1}{2} + 2 \log \varepsilon \frac{d}{r}\right) \text{ centimetres.}$$

The harmonic impedance factor for copper wires is

$$f = \sqrt{1 + \frac{p^2\lambda^2}{r^2}}$$

where $p = 2\pi n$

$r =$ the resistance of the wire per linear cm.

$$= \frac{\rho}{\pi r^2}$$

$$\therefore f = \sqrt{1 + \frac{p^2\pi^2 r^4\left(0.5 + 2 \log \varepsilon \frac{d}{r}\right)^2}{\rho^2}}.$$

This is the formula from which the Tables I. to VIII. have been computed, ρ being taken for 15°C as 1688.6 units C.G.S.

For any small change $d\rho$ in ρ, such as might be due to a change in assumed temperature or conductivity, the corresponding discrepancy introduced into the tubular value of the factor f is

$$df = -\frac{d\rho}{o}\left(\frac{f^2 - 1}{f}\right)$$

and the fractional error

$$\frac{df}{f} = -\frac{d\rho}{\rho} \cdot \left(\frac{f^2 - 1}{f^2}\right).$$

The coefficient

$$\frac{f^2 - 1}{f^2} \quad \text{or} \quad \left(1 - \frac{1}{f^2}\right)$$

is the coefficient of reduction from a small percentage variation in the resistance to the corresponding percentage variation produced in the impedance factor.

Thus for

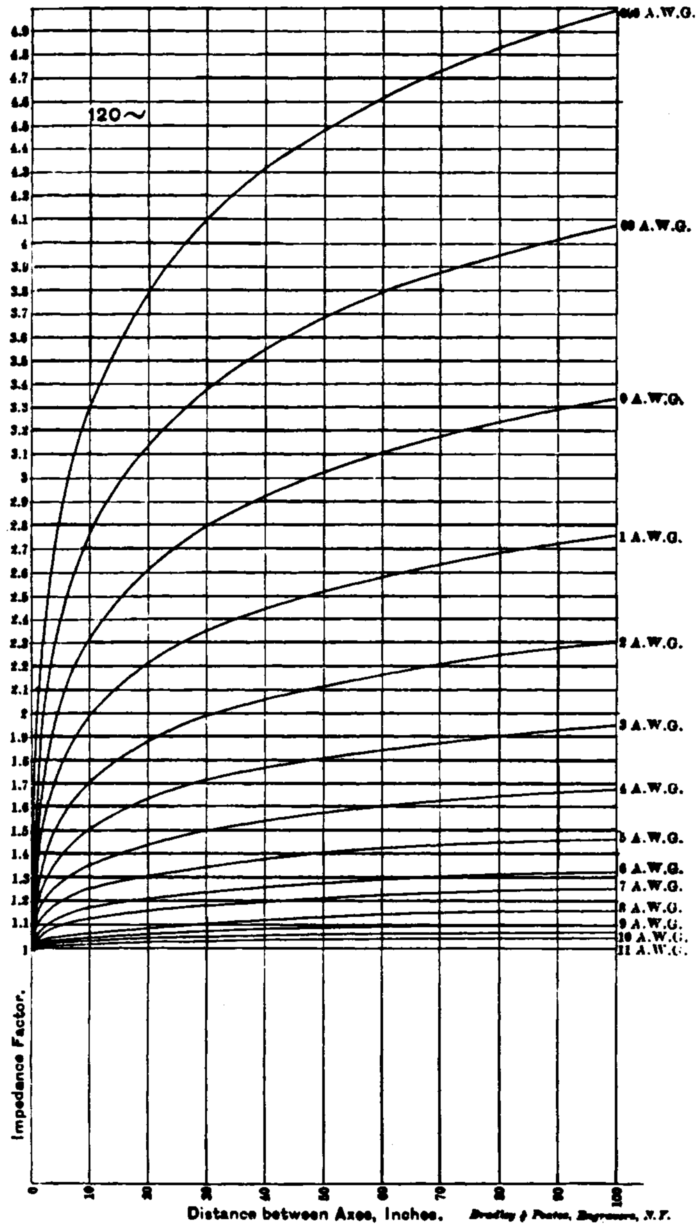
$$f = 2, \qquad \left(1 - \frac{1}{f^2}\right) = \frac{8}{4},$$

and a change of 4% additional resistivity in ρ will reduce the impedance factor by $4 \times \frac{3}{4}$ or 3%.

Particulars of the Measured Values of Impedance Factor on Three Parallel Overhead Copper Wires.

These parallel copper wires each 4,000 feet long and No. 6 B.W.G. 0.203" (0.516 cm.) in diameter, suspended on poles at an elevation of about 18 feet. The mean distance of these wires from one another is approximately one foot (30.5 cms.)

The mean resistance of each wire was 1.03 ω and the mean inductance of each wire when tested in looped pairs 1.264



Curve-Plate VII.

millihenrys, the temperature of the air in the shade being 29°C. Insulation of each wire approximately 30,000 ω to ground.

A 1,000-volt alternator of 141.6 ∼ whose E.M.F. is practically sinusoidal, was connected to a transformer reducing to 50 volts and this secondary pressure through volt and ammeters to successive looped pairs of these overhead wires. The mean impedance of each wire by volt ampere reading was 1.513 ω. Reproducing with the same instruments the same readings of current and voltage from a continuous current dynamo, the mean resistance was 1.029 ω per wire, making the observed impedance factor 1.47. With the mean measured inductance of 1.264 millihenrys the calculated impedance factor is 1.48 by the formula

$$f = \sqrt{1 + \frac{p^2 l^2}{r^2}}\,.$$

For No. 6 B.W.G. wires, the factor for an interaxial distance of 12 inches is in fact 1.485 by interpolation from the curves and tables.

APPENDIX II.

If the periodic current of c amperes (effective) flowing through a wire at n ∼ be analyzed into harmonic components:—

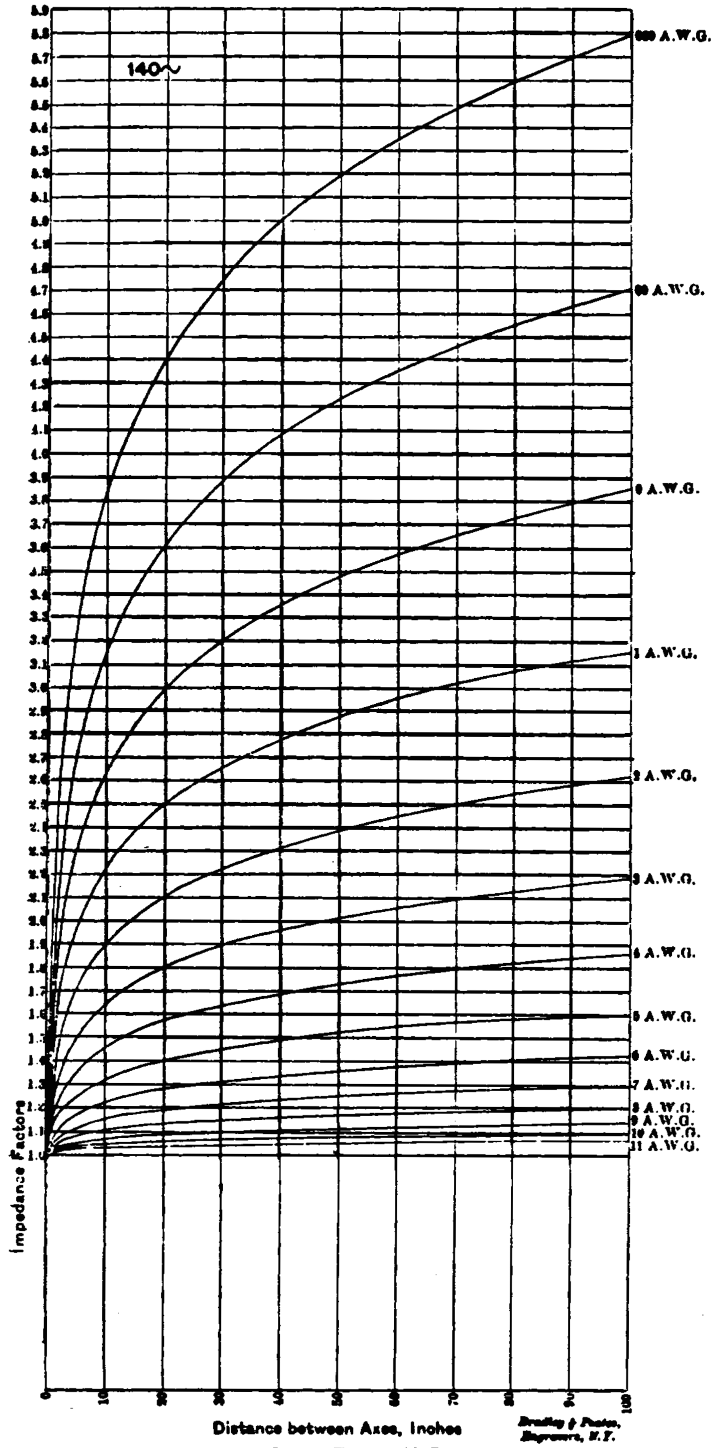
$$\begin{aligned} c &= C \sin pt + D \sin(2pt + \theta_2) + E \sin(3pt + \theta_3) \\ &\quad + F \sin(4pt + \theta_4) + \\ &= C\{\sin pt + \alpha \sin(2pt + \theta_2) + \beta \sin(3pt + \theta_3) \\ &\quad + \gamma \sin(4pt + \theta_4) + \end{aligned}$$

TABLE VII.

IMPEDANCE FACTORS FOR PAIRS OF PARALLEL COPPER WIRES CARRYING SIMPLE HARMONIC CURRENTS AT 120 AND 140 ~ 15°C.

$\frac{D}{r}$	No. 11, A. W. Gauge. Dia. 0.09074 in. 0.23045 cm Rad. 0.04537 in. 0.1152 cm			No. 10, A. W. Gauge. Dia. 0.10190 in. 0.2588 cm Rad. 0.05095 in. 0.1294 cm			No. 9, A. W. Gauge. Dia. 0.1144 in. 0.2906 cm Rad. 0.0572 in. 0.1453 cm			No. 8, A. W. Gauge. Dia. 0.1285 in. 0.3263 cm Rad. 0.0643 in. 0.1632 cm		
	D. (inches.)	factor. 120 ~	140 ~	D. (inches.)	factor. 120 ~	140 ~	D. (inches.)	factor. 120 ~	140 ~	D. (inches.)	factor. 120 ~	140 ~
2	0.0907	1.0006	1.0008	0.102	1.001	1.0013	0.114	1.0016	1.0021	0.1285	1.0025	1.0037
3	0.136	1.0013	1.0017	0.153	1.002	1.003	0.172	1.0032	1.0043	0.193	1.005	1.007
4	0.181	1.0019	1.0025	0.204	1.003	1.006	0.229	1.0047	1.0063	0.257	1.0075	1.010
5	0.227	1.0024	1.0033	0.255	1.004	1.005	0.286	1.0065	1.0080	0.321	1.0096	1.013
6	0.272	1.0029	1.0039	0.306	1.005	1.006	0.343	1.0073	1.010	0.386	1.212	1.015
8	0.363	1.0038	1.0051	0.408	1.006	1.008	0.458	1.0095	1.013	0.514	1.015	1.021
10	0.454	1.0045	1.0062	0.510	1.007	1.010	0.572	1.011	1.016	0.643	1.018	1.024
12	0.544	1.0052	1.0071	0.611	1.008	1.011	0.687	1.013	1.018	0.771	1.021	1.028
16	0.726	1.0063	1.0086	0.815	1.011	1.014	0.915	1.016	1.022	1.028	1.025	1.034
20	0.907	1.0073	1.0099	1.019	1.012	1.016	1.144	1.018	1.025	1.285	1.029	1.039
25	1.134	1.008	1.011	1.274	1.013	1.018	1.431	1.021	1.028	1.606	1.033	1.045
30	1.361	1.009	1.012	1.529	1.015	1.020	1.717	1.023	1.031	1.928	1.037	1.050
40	1.815	1.011	1.014	2.038	1.017	1.023	2.288	1.027	1.036	2.570	1.043	1.057
60	2.723	1.013	1.018	3.057	1.021	1.028	3.433	1.033	1.044	3.855	1.052	1.070
80	3.63	1.015	1.020	4.076	1.024	1.032	4.577	1.037	1.050	5.141	1.058	1.078
100	4.537	1.016	1.022	5.095	1.026	1.035	5.72	1.041	1.055	6.425	1.064	1.086
150	6.806	1.019	1.026	7.642	1.030	1.041	8.58	1.048	1.064	9.638	1.075	1.100
200	9.074	1.021	1.029	10.19	1.033	1.045	11.44	1.053	1.071	12.85	1.083	1.111
300	13.61	1.025	1.033	15.29	1.039	1.052	17.17	1.060	1.081	19.28	1.094	1.127
400	18.15	1.027	1.036	20.38	1.042	1.057	22.88	1.066	1.089	25.7	1.103	1.139
500	22.69	1.029	1.039	25.48	1.045	1.061	28.61	1.071	1.095	32.13	1.111	1.148
600	27.23	1.030	1.041	30.57	1.048	1.064	34.33	1.075	1.100	38.55	1.116	1.156
700	31.76	1.031	1.043	35.66	1.050	1.067	40.05	1.078	1.105	44.98	1.122	1.162
800	36.3	1.032	1.045	40.76	1.052	1.070	45.77	1.081	1.109	51.41	1.126	1.168
900	40.83	1.034	1.046	45.85	1.054	1.072	51.49	1.084	1.112	57.82	1.130	1.174
1,000	45.37	1.035	1.047	50.95	1.055	1.074	57.21	1.086	1.115	64.25	1.134	1.178
1,500	68.06	1.039	1.053	76.42	1.061	1.082	85.82	1.096	1.128	96.38	1.149	1.198
2,000	90.74	1.042	1.056	101.9	1.066	1.089	114.4	1.103	1.138	128.5	1.159	1.212
3,000	136.1	1.046	1.063	152.9	1.073	1.098	171.7	1.113	1.151	192.8	1.175	1.231
4,000	181.5	1.049	1.066	203.8	1.078	1.104	228.8	1.121	1.161	257.1	1.186	1.246
5,000	226.8	1.052	1.070	254.7	1.081	1.109	286.1	1.127	1.169	321.3	1.195	1.258
10,000	453.7	1.060	1.082	509.5	1.095	1.127	572.1	1.147	1.196	642.8	1.225	1.297

$\frac{D}{r}$	No. 7, A. W. Gauge. Dia. 0.1443 in. 0.3665 cm Rad. 0.07214 in. 0.1832 cm			No. 6, A. W. Gauge. Dia. 0.1620 in. 0.4115 cm Rad. 0.07214 in. 0.1832 cm			No. 5, A. W. Gauge. Dia. 0.1819 in. 0.4621 cm Rad. 0.0910 in. 0.231 cm		
	D. (inches.)	factor. 120 ~	140 ~	D. (inches.)	factor. 120 ~	140 ~	D. (inches.)	factor. 120 ~	140 ~
2	0.1443	1.0039	1.0053	0.162	1.0063	1.009	0.1819	1.010	1.014
3	0.2164	1.008	1.011	0.243	1.0128	1.017	0.273	1.020	1.027
4	0.2886	1.012	1.016	0.324	1.019	1.025	0.364	1.030	1.040
5	0.3608	1.018	1.021	0.4051	1.024	1.032	0.455	1.038	1.051
6	0.4329	1.0207	1.025	0.4861	1.029	1.040	0.546	1.046	1.062
8	0.5772	1.024	1.030	0.6482	1.038	1.051	0.728	1.059	1.080
10	0.7214	1.029	1.038	0.8101	1.045	1.061	0.9095	1.071	1.095
12	0.8658	1.033	1.044	0.9721	1.052	1.069	1.091	1.081	1.108
16	1.154	1.040	1.054	1.296	1.063	1.084	1.455	1.098	1.131
20	1.443	1.046	1.062	1.621	1.072	1.096	1.819	1.112	1.149
25	1.804	1.052	1.070	2.025	1.082	1.109	2.274	1.127	1.169
30	2.164	1.058	1.077	2.430	1.090	1.120	2.728	1.140	1.186
40	2.886	1.067	1.090	3.241	1.104	1.139	3.638	1.061	1.214
60	4.329	1.080	1.108	4.861	1.125	1.167	5.457	1.193	1.255
80	5.772	1.091	1.122	6.482	1.141	1.188	7.276	1.217	1.286
100	7.214	1.099	1.133	8.101	1.154	1.205	9.095	1.236	1.311
150	10.82	1.116	1.155	12.15	1.179	1.237	13.64	1.273	1.358
200	14.43	1.128	1.171	16.21	1.197	1.261	18.19	1.300	1.393
300	21.64	1.146	1.195	24.30	1.225	1.296	27.28	1.340	1.443
400	28.86	1.160	1.212	32.41	1.245	1.322	36.38	1.369	1.480
500	36.08	1.171	1.226	40.51	1.261	1.342	45.48	1.392	1.508
600	43.29	1.180	1.238	48.61	1.274	1.359	54.57	1.411	1.532
700	50.50	1.187	1.248	56.71	1.285	1.374	63.67	1.427	1.553
800	57.72	1.194	1.257	64.82	1.295	1.387	72.76	1.442	1.571
900	64.93	1.200	1.265	72.91	1.304	1.398	81.86	1.454	1.587
1,000	72.14	1.206	1.272	81.01	1.312	1.408	90.95	1.466	1.602
1,500	108.2	1.228	1.300	121.5	1.344	1.448	136.4	1.510	1.657
2,000	144.3	1.244	1.321	162.0	1.367	1.477	181.9	1.543	1.698
3,000	216.5	1.267	1.350	243.0	1.400	1.519	272.9	1.590	1.755
4,000	288.6	1.283	1.372	324.0	1.424	1.550	363.8	1.624	1.797
5,000	360.7	1.297	1.388	405.1	1.443	1.573	454.8	1.650	1.829
10,000	721.4	1.341	1.444	810.2	1.506	1.652	909.5	1.737	1.936



Curve-Plate VIII.

The impedance factor for the wire will be

$$f=\sqrt{\frac{\left(1+\frac{p^2l^2}{r^2}\right)+\alpha_2\left(1+\frac{2^2p^2l^2}{r^2}\right)+\beta^2\left(1+\frac{3^2p^2l^2}{r^2}\right)+\gamma^2\left(1+\frac{4^2p^2l^2}{r^2}\right)+}{1+\alpha_2+\beta^2+\gamma^2+}}$$

or if $\frac{p^2l^2}{r^2}=J$

$$f=\sqrt{\frac{(1+J)+\alpha_2(1+4J)+\beta^2(1+9J)+\gamma^2(1+16J)+}{1+\alpha_2+\beta^2+\gamma^2+}}\,.$$

TABLE VIII.

IMPEDANCE FACTORS FOR PAIRS OF PARALLEL COPPER WIRES CARRYING SIMPLE HARMONIC CURRENTS OF 120 ~ AND 140 ~ AT 15°C.

$\frac{D}{r}$	No. 4. A. W. Gauge. Dia. 0.2043 in. 0.5189 cm Rad. 0.1022 in. 0.2595 cm			No. 3. A. W. Gauge. Dia. 0.2294 in. 0.5827 cm Rad. 0.1147 in. 0.2914 cm			No. 2. A. W. Gauge. Dia. 0.2576 in. 0.6543 cm Rad. 0.1288 in. 0.3272 cm			No. 1. A. W. Gauge. Dia. 0.2893 in. 0.7344 cm Rad. 0.1447 in. 0.3672 cm		
	D. (inches.)	factor. 120 ~	140 ~	D. (inches.)	factor. 120 ~	140 ~	D. (inches.)	factor. 120 ~	140 ~	D. (inches.)	factor. 120 ~	140 ~
2	0.2043	1.016	1.022	0.2294	1.025	1.034	0.2576	1.039	1.053	0.2893	1.062	1.083
3	0.3065	1.032	1.043	0.3441	1.051	1.068	0.3864	1.079	1.106	0.4339	1.123	1.164
4	0.4086	1.047	1.063	0.4588	1.073	1.098	0.5153	1.114	1.153	0.5785	1.176	1.234
5	0.5108	1.060	1.080	0.5735	1.094	1.1255	0.6441	1.145	1.185	0.7232	1.223	1.294
6	0.613	1.072	1.096	0.6882	1.112	1.150	0.7730	1.173	1.230	0.8678	1.264	1.347
8	0.8172	1.092	1.124	0.9177	1.143	1.191	1.030	1.220	1.291	1.157	1.333	1.435
10	1.0215	1.110	1.147	1.147	1.170	1.226	1.288	1.260	1.338	1.447	1.391	1.507
12	1.226	1.1255	1.167	1.376	1.193	1.256	1.546	1.294	1.385	1.736	1.439	1.568
16	1.634	1.151	1.201	1.835	1.232	1.306	2.061	1.351	1.457	2.314	1.520	1.668
20	2.043	1.173	1.229	2.294	1.264	1.347	2.576	1.396	1.514	2.893	1.584	1.748
25	2.554	1.195	1.259	2.868	1.297	1.389	3.221	1.444	1.574	3.616	1.651	1.826
30	3.065	1.214	1.283	3.441	1.325	1.424	3.864	1.484	1.624	4.339	1.706	1.898
	4.086	1.246	1.324	4.588	1.371	1.482	5.153	1.549	1.704	5.785	1.796	2.007
60	1.13	1.294	1.384	6.882	1.439	1.567	7.730	1.644	1.821	8.678	1.925	2.164
80	8.172	1.328	1.429	9.177	1.480	1.630	10.30	1.713	1.906	11.57	2.019	2.277
100	10.215	1.357	1.464	11.471	1.528	1.679	12.88	1.768	1.973	14.47	2.093	2.366
150	15.32	1.409	1.531	17.21	1.603	1.771	19.32	1.870	2.097	21.70	2.229	2.530
200	20.43	1.448	1.579	22.94	1.657	1.837	25.76	1.943	2.185	28.93	2.326	2.647
300	30.65	1.505	1.649	34.41	1.735	1.933	38.64	2.049	2.313	43.39	2.466	2.813
400	40.86	1.546	1.700	45.88	1.791	2.001	51.53	2.124	2.404	57.85	2.566	2.932
500	51.08	1.578	1.740	57.35	1.836	2.055	64.41	2.184	2.476	72.32	2.644	3.026
600	61.30	1.605	1.773	68.82	1.872	2.099	77.30	2.232	2.534	86.78	2.708	3.101
700	71.50	1.627	1.801	80.30	1.903	2.138	90.16	2.274	2.584	101.22	2.762	3.166
800	81.72	1.648	1.826	91.77	1.931	2.171	103.0	2.310	2.627	115.7	2.810	3.222
900	91.92	1.666	1.848	103.2	1.955	2.200	115.9	2.342	2.666	130.1	2.852	3.272
1000	102.15	1.681	1.867	114.7	1.976	2.226	128.8	2.370	2.700	144.7	2.889	3.316
1500	153.3	1.743	1.943	172.1	2.060	2.327	193.2	2.482	2.832	217.0	3.034	3.487
2000	204.3	1.788	1.998	229.4	2.120	2.400	257.6	2.561	2.927	289.2	3.131	3.610
3000	306.5	1.852	2.076	344.2	2.206	2.502	386.5	2.674	3.061	434.0	3.283	3.782
4000	408.6	1.898	2.132	458.9	2.267	2.575	515.3	2.754	3.156	578.5	3.385	3.904
5000	510.8	1.934	2.175	573.5	2.315	2.633	644.2	2.816	3.230	723.3	3.467	3.999
10000	1021.5	2.051	2.317	1147.1	2.471	2.819	1288.	3.019	3.471	1447.	3.729	4.308

$\frac{D}{r}$	No. 0. A. W. Gauge. Dia. 0.3249 in. 0.8252 cm Rad. 0.1624 in. 0.4126 cm			No. 00. A. W. Gauge. Dia. 0.3648 in. 0.9265 cm Rad. 0.1824 in. 0.4628 cm			No. 000. A. W. Gauge. Dia. 0.4096 in. 1.040 cm Rad. 0.2048 in. 0.520 cm		
	D. (inches.)	factor. 120 ~	140 ~	D. (inches.)	factor. 120 ~	140 ~	D. (inches.)	factor. 120 ~	140 ~
2	0.3249	1.096	1.130	0.3648	1.150	1.199	0.4096	1.230	1.303
3	0.4873	1.189	1.251	0.5473	1.288	1.377	0.6144	1.431	1.558
4	0.6498	1.269	1.353	0.7296	1.404	1.523	0.8192	1.595	1.761
5	0.8122	1.337	1.440	0.912	1.501	1.645	1.024	1.730	1.927
6	0.9747	1.397	1.515	1.094	1.585	1.748	1.229	1.845	2.066
8	1.300	1.496	1.638	1.459	1.722	1.917	1.638	2.032	2.293
10	1.6243	1.576	1.739	1.824	1.833	2.053	2.048	2.181	2.472
12	1.950	1.645	1.823	2.189	1.926	2.166	2.458	2.305	2.621
16	2.599	1.755	1.958	2.919	2.076	2.346	3.276	2.503	2.857
20	3.249	1.844	2.066	3.648	2.194	2.489	4.096	2.659	3.043
25	4.061	1.935	2.176	4.560	2.315	2.633	5.120	2.817	3.231
30	4.873	2.010	2.267	5.473	2.415	2.756	6.144	2.947	3.385
	6.498	2.130	2.411	7.296	2.574	2.941	8.192	3.153	3.629
60	9.747	2.303	2.619	10.94	2.80	3.211	12.29	3.447	3.976
80	13.00	2.427	2.767	14.59	2.962	3.403	16.38	3.655	4.222
100	16.24	2.524	2.884	18.24	3.089	3.553	20.48	3.818	4.415
150	24.37	2.703	3.096	27.36	3.320	3.828	30.72	4.117	4.764
200	32.49	2.831	3.248	36.48	3.486	4.023	40.96	4.329	5.014
300	48.73	3.014	3.464	54.73	3.722	4.299	61.44	4.629	5.368
400	64.98	3.143	3.618	72.96	3.888	4.496	81.92	4.842	5.619
500	81.22	3.243	3.737	91.20	4.019	4.649	102.4	5.009	5.812
600	97.47	3.327	3.835	109.4	4.124	4.773	122.9	5.144	5.971
700	113.7	3.397	3.918	127.7	4.214	4.880	143.3	5.258	6.106
800	130.0	3.458	3.991	145.9	4.294	4.972	163.8	5.359	6.203
900	146.2	3.513	4.055	164.1	4.362	5.054	184.3	5.446	6.327
1000	162.4	3.561	4.111	182.4	4.424	5.125	204.8	5.525	6.418
1500	243.7	3.747	4.330	273.6	4.662	5.405	307.2	5.828	6.773
2000	324.9	3.879	4.487	364.8	4.832	5.604	409.6	6.044	7.072
3000	487.4	4.067	4.708	547.8	5.071	5.884	614.4	6.348	7.381
4000	649.8	4.200	4.863	729.6	5.240	6.140	819.2	6.563	7.634
5000	812.2	4.304	4.985	912.0	5.371	6.237	1024.	6.730	7.828
10000	1624.	4.638	5.379	1824.	5.798	6.737	2048.	7.271	8.462

The primary current wave observed by Messrs. Ryan and Merritt as supplied to an unloaded transformer is approximately compounded as follows. See also p. 454 Fleming's "Transformer," vol. ii.

$$i = 0.1963 \sin(pt - 49°.20') + 0.04827 \sin(3pt - 76°.50') + 0.01627 \sin(5pt - 90°).$$

Here $\beta = \frac{0.04827}{0.1963} = 0.2463$, and $o = \frac{0.01627}{0.1963} = 0.08156$.

$$\therefore f = \sqrt{\frac{\left(1 + \frac{p^2l^2}{r^2}\right) + 0.06066\left(1 + 9\frac{p^2l^2}{r^2}\right) + 0.006652\left(1 + 25\frac{p^2l^2}{r^2}\right)}{1 + 0.06066 + 0.006652}}$$

$$= \sqrt{1 + 1.605\frac{p^2l^2}{r^2}}$$

$$= \sqrt{1 + 63.35\frac{n^2l^2}{r^2}}.$$

DISCUSSION.

THE CHAIRMAN:—I think every one here will agree with me that the Institute is to be congratulated upon this magnificent paper of Mr. Kennelly's. It is indicative of a very large amount of painstaking care. There are quite a large number of gentlemen here in the meeting to-night who have made considerable study of this subject, and the Institute will be very glad to hear from them. Before, however, asking for remarks, the Chair would like to refer to one point in Mr. Kennelly's paper that he made inquiry about in London during the last year. Mr Kennelly referred to the "well-known Ferranti effect." As I understand, shortly after the station started—the sub-station, I think it is in Grosvenor Gallery, some miles distant from the main station, was supplied with electricity from the main station, and when tests were made, it was found, to the surprise of a great many people, that the potential, instead of being lower at that point, was higher than it was at the original source.

Last year while in London, I visited the Ferranti station at Deptford a number of times and inquired particularly of Mr. d'Alton, the manager and electrician of the company, whether this so-called Ferranti effect existed upon the Ferranti mains, and whether it was due to an electrostatic effect or not. He informed me that it did not exist, and that many prominent scientific men had rushed into print to explain theoretically the causes of this effect; that a number of papers had been written about it, but that it was found afterwards to be an effect which did not exist. I asked if he could show me the difference of potential between the generating station and the sub-station, and he took me to the switchboard and stated that the difference of potential was several volts lower at the sub-station than at the generating station. Periodically a signal was sent from the latter to the former, indicating that the potential had fallen below normal. "If you will stand by the instrument," he said, "you will see that the difference of potential is exactly what the drop is calculated to be between the sub-station and the main-station and it is higher at the original generating source than at the other. Accompanying me at the time was Mr. C. H. W. Biggs, the editor of the *Electrical Engineer* of London. Mr. Biggs and another editor of a paper in London stated to me that this Ferranti effect did not exist, and that a great deal had been written about it that should not have been written. Mr. Biggs said that at the time this appeared, he stated editorially that it was based upon wrong premises. This is all that I can say personally about it, but I doubt not that Mr. Kennelly can give further information on the subject and I should be pleased to hear from him upon the subject.

MR. DOUGLASS BURNETT:—Mr. Chairman, let us for a moment consider a little more carefully two of Mr. Kennelly's equations. First, the one on p. 178.[7] Let us put it equal to c; i.e.,

$$c = epk = 2\pi ekn, \tag{I.}$$

and regard c and n as the variables. It is evident that the curve is a straight line passing through the origin. In other words, for zero frequency, we get zero current through the condenser, and thereafter current is directly proportional to frequency. For infinite frequency, infinite current passes, at any finite voltage e.

Second observe the formula of p. 179,[7] for impedance. We may put it equal to I, and hence

$$c = \frac{e}{I} = \frac{e}{\sqrt{r^2 + \left(\frac{1}{pk}\right)^2}} = \frac{e}{\sqrt{r^2 + \left(\frac{1}{2\pi kn}\right)^2}}. \tag{II.}$$

This may be transformed into

$$n = \frac{1}{2\pi k + \sqrt{\frac{e^2}{c^2} - r^2}}. \tag{III.}$$

We see directly that:

(a) For the same frequency, current is directly proportional to E.M.F., which was to be expected; that

(b) For the same E.M.F., resistance, and capacity, current is a complicated function of frequency; and that

(c) For the same current, necessary capacity is inversely proportional to frequency. Substituting, as in the paper,

$$\varepsilon = 1000; \quad r = 500; \quad k = 5 \times 10^{-6};$$

in either II. or III., we get the curve of frequencies and currents. [See Fig. 13, p. 232.][8] For direct currents,

$$n = 0 \,\&\, c = o. \quad \text{For } n = \infty, \quad c = \frac{e}{r} = \frac{1000}{500} = 2.$$

For frequencies above 200, c lies between 1.906 and 2.0. This last result is the same as if the condenser were quite short-circuited, and only the resistance existed in circuit.

[7]TRANSACTIONS, vol. x, 1893. This issue, p. 460.
[8]TRANSACTIONS, vol. x, 1893. This issue, p. 487.

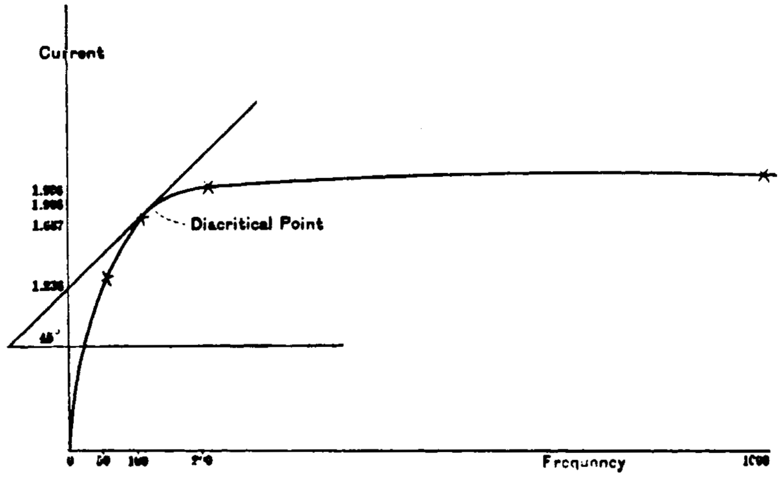


Fig. 13. (See Burnett's Remarks, p. 218.)

Equations II. and III. probably apply to (1) the leakage current of a condenser, and to (2) an electrolytic cell used as a condenser, especially when current-density on electrodes is smaller than that value at which decomposition takes place.

If we consider as the best in practice, a diacritical point on the frequency-current curve (see Fig. 13), as with **H–B** curves, at which the tangent to the curve makes equal angles with both axes, we should have (from III.)

$$\frac{dc}{dn} = 1. \qquad \text{(IV.)}$$

From which we find that

$$\varepsilon^2 = 2\pi k(\varepsilon^2 - c^2r^2)^{3/2} \qquad \text{(V.)}$$

is the best condition. That is, in designing apparatus for a circuit consisting of condenser and resistance in series, such a frequency should be chosen as makes the different quantities have the relation given by V.

It is to be noticed that the laws given by Mr. Kennelly are strictly applicable to Wheatstone's Bridge.

While on the relations between frequency and other quantities, one effect may be worth mentioning. We consider, with Fleming,[9] a circuit of resistance R and inductance L, carrying current x, in parallel with a similar one of resistance S and inductance N, carrying current y; then when $SL > RN$, the current y leads ahead of the main current by an angle φ, determined by

$$\tan\varphi = \frac{(LS - RN)p}{R(R+S) + L(L+N)p^2}.$$

Here for both $p = 0$ and $p = \infty$, φ becomes 0. That is, for continuous and for infinitely rapid currents, there is no lead. Using the ordinary rule of calculus for determining the maximum, we find that for

$$n = \frac{1}{2\pi}\sqrt{\frac{R(R+S)}{L(L+N)}} \qquad \text{(VI.)}$$

the function is a maximum. Or,

$$\tan\varphi = \frac{SL - RN}{2\sqrt{RL(R+S)(L+N)}}. \qquad \text{(VII.)}$$

Therefore, for a certain determinable frequency, dependent on the resistances and inductances, the lead is a maximum.

These points are mentioned as showing the importance of remembering that in discussions like Mr. Kennelly's, we are dealing with a factor often subordinated, but which sometimes exerts great influence—namely, frequency.

Mr. T. D. Lockwood:—I presume that each member present had been profoundly interested in Mr. Kennelly's paper, and that each one of us can also say, and say it sincerely, that this evening we have added something to our stock of knowledge—not so much, perhaps, that we can all assimilate at once everything we have heard, because the paper, though beautifully plain, and made still more plain by the admirable extemporaneous exposition which accompanied it, is still far too copious and exhaustive to be readily digested at one sitting; and this I conceive, Mr. Chairman, is the real value of the paper, that we have had a store of knowledge, as it were, bought right to us, and so placed that it can be embodied in our proceedings in order that at any time we may refresh our memory and replenish our minds by going to it, as we now go to a text-book or an encyclopaedia.

It is now, I think, three years since at the Boston meeting of this Institute,[10] I pointed out that impedances were capable of, and had already been employed in many useful applications to electric work, although we did not at that time often make use of the word "impedance." It seems to me that one of the great beauties of practical electrical work is that functions, magnitudes, and apparently disadvantageous phenomena, which seem at first to be prejudicial to good work, can often be utilized and made subservient to good work in other directions. I may instance the polarization, which to the early battery inventors and constructors was such an obstacle, frequently opposing their best attempts at success; yet at the present time the phenomena of polarization having been more fully worked out, we find that it can be made really advantageous, and that by prosecuting researches in such phenomena, the storage battery has been reached; so, taking my own special line of work, it was found in the very early years of telephony that an electro-magnet in a telephonic circuit was a very strong bar to the passage of the telephonic impulses; so much so that one of our members, Mr. F. W. Jones, in devising a remedy for it, said that electro-magnets were *opague* to rhythmical or rapidly alternating currents. But with the greater advance of the science in its many industrial applications, and our own improved knowledge and experience, it is found that impedances can actually be made useful, in that

[9] Fleming; "Alternate Current Transformer," vol. i, p. 133.

[10] "The Industrial Utilization of the Counter Electromotive Force of Self-Induction" Transactions, vol. vii., p. 226.

two wires can be made use of to transmit three messages at the same time, using the two wires as the direct and return wires of the metallic telephone circuit; while on the Van Rysselberghe plan, each one of the two wires composing that circuit may be severally used for the transmission of telegraphic messages; and this simply by placing impedances, made by properly winding covered wire around the iron, and interposing them between the telephonic circuit and the telegraph circuit proper. Such contrivances are much older however in telegraphy. I believe Mr. Cromwell Fleetwood Varley (now gone from us, but whom England and America delighted to honor,) used impedances in 1870—he called them echocymes —for a similar purpose, that is for working Morse telegraphy and harmonic telegraphy on one and the same wire. In the matter of electric lighting we find that Professor John Hopkinson devised impedances by winding wires, the amount of which could be made adjustable, on closed magnetic circuit laminated iron cores, in order that arc lights might be regulated in multiple arc; and I need scarcely call the attention of any one here to the wonderful application in the automatic regulation of lighting by transformers, as suggested, I believe, by Mr. Kennedy, but worked out by his successors, Messrs. Zipernowski, Déri and Blathy.

I have mentioned that "impedance" is an extremely modern term, it having been devised and given to us by Mr. Oliver Heaviside, within the last seven years; it is, however, a very graphic and highly suitable term, and I for one am much obliged to Mr. Oliver Heaviside for this favor, and others of the same kind.

Mr. Chairman, I have heard or read, I cannot tell which, at the present time, that the late Lord Derby in one of his addresses to students, defined knowledge as that which a man has so secured that he can produce it at any moment from his own head. Referring once more to the remarks I first made, I would like to say that I agree with Mr. John T. Sprague entirely, in his view that while that definition is no doubt a first class definition when cramming for an examination, it is not such a good definition when applied to the man of science who must use his science to earn his daily bread; and that real knowledge, the knowledge which the practical man carries in his head, is not perhaps the most valuable part of his knowledge. What really constitutes the best part of his outfit, is the knowing where to find what he wants when he wants it; and it seems to me that the paper we have had this evening will be a great aid to many of us in that kind of knowledge. Hereafter when we want to know anything about impedance, so far at least as it has up to the present time been investigated, we shall come to Mr. Kennelly's paper.

The knowledge which we can carry in our heads is, it seems to me, very much like money we carry in our pockets—very useful for making small change at the moment; but not so useful to carry on business with. But the knowledge which we can get hold of when we need it, when we are asked a question about it, or when a question comes up in our business—the knowledge that we can have recourse to, and find out what we want—such knowledge is more like money that we have in our bank accounts, that is to say, if we have bank accounts, and any money in them.

Dr. Pupin:—Mr. Chairman, I intended to make several remarks on the paper, but since it is so late I feel that I must cut them down and be very brief.

In the first place, it strikes me that Mr. Kennelly has done very good service to practical electrical engineers in making tables to which the electrical engineer can always refer, just as he refers to tables for his resistances, and finds out what will be the impedance of such and such a circuit. It is a very useful thing and it is done in the way in which only Mr. Kennelly can do it. He is well known for his neatness, and for his patience in working out such problems, so that any eulogistic remarks from me would be superfluous.

The other point which struck me as a very good point indeed, is Mr. Kennelly's remark regarding the simplicity of the alternating current theory. I pointed out in a paper read three years ago at the Boston meeting of this Institute,[11] that the alternating current theory is just as simple as the continuous current theory, it is based on one single law, namely, Ohm's law. But, of course; we have to limit ourselves to instantaneous values of things, and just as we say in a continuous current circuit, that the impressed electromotive force plus the total drop is equal to zero, so we can say in the alternating current circuit that the sum of all the electromotive forces taken with their proper sign plus the total drop is equal to zero. We have therefore one and the same law for both kinds of current. That gives us our fundamental equation or fundamental relation between the quantities which are involved. But since that relation is true for infinitely short intervals of time, its symbolical expression gives us what is called in mathematics a different equation. The integral relations which will hold true for any interval of time are obtained by the process of integration. Now in finding the integral relation between the current and the other quantities which define the circuit, like self-induction, capacity, resistance, etc., we find that if we want to obtain the current, we have to divide the impressed E.M.F., not by the resistance alone, but by the resistance plus something else, and that something else is composed with the resistance, just the same way as two forces, namely by the parallelogram of forces—two forces which are at right angles to each other. We have therefore the simple rule that the resistance can be composed with the inductance speed, as Mr. Kennelly calls it, just the same as one force can be composed with another which is at right angles to it. In other words the parallelogram of forces is applicable to this case. If there is a condenser, there is capacity in the circuit, and in finding the integral relation between the current and the time, we find that the behavior of the condenser can be represented graphically in a very simple way by introducing into the parallelogram of forces just mentioned, a third component equal to the capacity speed, this third component to be always subtracted from the component representing inductance speed.

I call this graphic method of representing impedance the application of the parallelogram of forces to the alternating current circuits. Mr. Kennelly prefers to speak of vector quantities and the addition of vectors. But, of course, these graphical methods are incidental results of considerable practical importance. The primary law is Ohm's law in its generalized form, and its applicability to variable currents, variable but stationary. If the flow is not stationary, then Ohm's law even in this generalized form will be applicable to infinitely short lengths only of linear conductors as, for instance, in the case of very long wires and high frequency.

The next point which I wish to discuss very briefly is the so called Ferranti effect. I am, however, somewhat timid about it after the remarks of our Chairman, saying that a great many people had rushed into print about this Ferranti effect.

[11]"Practical Aspects of the Alternating Current Theory": Transactions, vol. vii, p. 204, 1890.

THE CHAIRMAN:—It was merely the statement that was given to me.

DR. PUPIN:—Well, the statement then that was given to our Chairman makes me hesitate. I also rushed into print about something like the Ferranti effect. I published a paper in the *American Journal of Science*[12] and the next paper is now in print about this very thing. The Ferranti effect is a real, existing effect and can be made very strong indeed. Most people have no idea how strong this effect can be made. It is very simple. The Ferranti effect is only a special case of the more general effect which I call the resonance. It could have been foreseen twenty-five years ago from Maxwell's equations deduced at about that period. Maxwell, I think, was the first to show the effect of introducing a condenser capacity into an alternating current circuit and it is very interesting to observe this circumstance. Maxwell was spending an evening with Sir William Grove who was then engaged in experiments on vacuum tube discharges. He used an induction coil for this purpose, and found that if he put a condenser in parallel with the primary circuit of his induction coil, that he could get very much larger sparks, which meant, of course, that he got a very much larger current through his primary coil, an alternating current generator being used to feed the primary. He could not see why. Maxwell, at that time, was a young man. That was about 1865, if I do not err. Grove knew that Maxwell was a splendid mathematician, and that he also had mastered the science of electricity as very few men had, especially the theoretical part of it, and so he thought he would ask this young man how it was possible to obtain such powerful currents in the primary circuit by adding a condenser. Maxwell who had not had very much experience in experimental electricity at that time, was at a loss. But he spent that night in working over his problem, and the next morning he wrote a letter to Sir William Grove explaining the whole theory of the condenser in multiple connection with a coil. It is wonderful what a genius can do in one night! He pointed out the exact relations between the condenser the self-induction and the frequency which would give the largest current, and he was the first to do this, so far as I know.

We must always remember that as soon as we add capacity to a circuit we are giving it elasticity. Without capacity the circuit has no elasticity. Take a stiff wire and suspend a weight, say a cylindrical bar, by it. Suppose that this wire has no elasticity. In order to twist this weight, in order to deflect it from its position of equilibrium, we have to use a force that will twist the weight out of shape. The moment of inertia of the weight, when twisted around, is being changed. That is just what happens in an alternating current circuit which has no appreciable capacity. The electromotive force working on such a circuit produces forced vibrations. But suppose that the stiff wire has elasticity. Then if you give an impulse to the weight, it will swing, and the period of the swing will depend on the moment of inertia of the weight, and on the elasticity of the wire and on nothing else, provided of course that the frictional resistances are not too large. We can make that weight swing rapidly by making the moment of inertia small or the elasticity large, one of the two, or both. Now the coefficient of self-induction in the circuit, corresponds to the moment of inertia of the swinging weight, and the capacity in the circuit corresponds to the elasticity of the wire, and just as the elasticity of the wire and the moment of inertia of the weight determine the period of the circuit, so the capacity in the circuit, and the self-induction determine the electrical period of the circuit. That is to say if you create a disturbance in the circuit, say by pulling quickly a permanent magnet away from the coil, you will start an electrical disturbance there, and the electricity will swing back and forth, just as a pendulum swings back and forth, the period of that swing depending on the coefficient of self-induction and the capacity of the circuit. The electricity in the circuit will swing back and forth till it is reduced to rest by the ohmic resistance. If you now apply a periodically varying impulse—not a single impulse, but a periodically varying impulse to the torsional pendulum just mentioned, you will make it oscillate, but the oscillation will be forced, if the period of the acting force is different from the natural period of the pendulum. But if the two periods are exactly the same, then the oscillation of the pendulum is a free oscillation, and the force and pendulum are in resonance. Now what is the effect of free oscillation? The effect of free oscillation is to reach a larger swing than the forced oscillation under the action of a force of the same mean intensity. The swing will continually increase, until the work done against the frictional resistances during one half of the swing is exactly equal to the work which the moving force does in that time. If, therefore, the resistance is very small you see that the swing will increase indefinitely. But what happens to the wire? Resonant swinging means simply this:—The kinetic energy of the swinging weight is periodically reduced to zero, that is, transformed entirely into the potential energy of the elastic forces of the wire and *vice versa*; so that the larger the swing, the larger will be the maximum elastic force with which the wire reacts; so that the smaller the frictional resistances the larger the elastic reaction. But we must remember that the elastic force in the wire corresponds to our potential difference in the condenser, so that by acting upon the circuit by means of an electromotive force whose period is exactly the same as the natural period of the circuit, we produce an electrical flow which continually increases until it has obtained a maximum value and at that time, of course, the potential difference in the condenser may be many times higher than the voltage of the impressed electromotive force. I have produced by very simple means a rise of potential from 60 volts to 900 volts just that way, and from 80 volts to nearly 1,200 volts by putting a condenser in series with a coil, and a proper adjustment of the two. I took the secondary of a transformer and placed with it a large coil, large self-induction, in series, and then, in series with this, a condenser; and then I adjusted my capacity to the self-induction in such a way that I brought the period of the circuit in resonance with the frequency of my alternator, and the consequence was, that the difference of potential (indicated by a Thomson electrostatic voltmeter) went from 80 volts to 1,200 volts.

If you create too much rise of potential your condenser goes. am working at that problem yet, and I expect to be able very soon, in fact I have already promised, to read a paper before the Institute on this very thing. I have no doubt that it is quite possible to transform 50 or 60 volts into 10,000 or 15,000 or any number of volts in that way.

On page 178 of the paper[13] it is mentioned that the reason for the non-equality of the voltage on the series of the two impedances with the arithmetical sum of the voltages measured on each impedance in turn, is due to the fact that the

[12] April, 1893.

[13] TRANSACTIONS, vol. x, 1893. This issue, p. 460.

currents in the two coils are usually out of step. I should like to point out that this is misleading. The current must have at any moment the same phase at every point of an alternating current circuit, otherwise there would be an accumulation or absorption of electricity at the points where a change of phase existed. There is evidently a *lapsus linguae* in the paragraph referred to, and I have no doubt that Mr. Kennelly will easily change the paragraph.

MR. KENNELLY:—Dr. Pupin is right concerning the paragraph on page 178,[13] which by an oversight is misleading. The sentence should read "due to the fact that 'otherwise' the currents in i_1 and i_2 while of equal strength develop in these counter E.M.F's that are out of step."

DR. GEYER:—Mr. Kennelly has in those plates given us very instructive information tending to show that in the alternating system of distribution there is a somewhat greater drop than in the continuous current system, and he has also called attention to the interesting fact that on account of the distortion of the sinusoidal wave by the transformer under light load, this extra drop is to some extent increased, so that the distribution is still less uniform than it would be without this action of the transformer. I believe all this applies to the primary part of the circuit. I should like to consider for a moment the secondary part of the circuit—say the house wiring. I do not at this moment recollect whether there is a corresponding distortion of the sinusoidal wave in the secondary part of the circuit, and I should like to ask Mr. Kennelly if there is or if there is not, a corresponding tendency there to increase the extra drop.

MR. KENNELLY:—As Dr. Geyer has remarked, the impedance factor increases above the tabular values when the current waves are not simply sinusoids, and in such cases the drop in the primary feeders is further augmented. But if the alternator supplying these feeders generates a wave of E.M.F. that is sinusoidal, the transformer must also generate E.M.F's that are very nearly sinusoidal, both in its primary and secondary coils, even although the primary current wave may be severely distorted under the influence of hysteresis in the core. Consequently, with only incandescent lamps, and no ferric inductances, in the secondary circuit, the secondary current waves will be very nearly sinusoidal, making the impedance factors of the secondary conductors practically tabular.

THE CHAIRMAN:—I want to say just a word in reference to Dr. Pupin's remark. The remarks that I made upon the Ferranti effect were based entirely upon the statements made to me by parties on the other side of the water, and referred particularly to the effect claimed to be produced upon the Ferranti cable, and not alone the peculiarity of a difference of potential which might be caused by electrostatic phenomena, and I should like to ask Mr. Kennelly's opinion as to the reliability of the reports respecting the Ferranti effect as produced upon his underground cable in London.

MR. KENNELLY:—Not having had the pleasure of being present at the measurements or observations that were made on the Ferranti cable, I am unable to give any direct evidence whatever, but having carefully read Dr. Fleming's very interesting paper, in which a large number of experiments were described upon the pressure as generated and supplied, I think there is no escape from the conclusion that under certain conditions of load and transmission, an effect was produced which bears the name of the Ferranti effect; but that possibly under conditions which were described by the Chairman that effect did not exist. It is entirely of course, a matter of the resonance in the circuit, as Dr. Pupin has described it, and if you do not have the right load that effect may disappear. So it is quite easy to understand that the cable, under certain conditions of load or transmission, might be resonant and might exalt the pressure; while under normal load this effect might cease to exist.

MR. C. S. BRADLEY:—Mr. Chairman, I think the paper is entirely beyond my scope. I should like to thoroughly understand it. It has given me a great many points and it is very clear indeed. I am very glad to have heard it, and I think it will prove of great value as a reference. I have not digested it yet, to borrow Mr. Lockwood's phrase, and therefore I am not prepared to say very much about it.

MR. LOCKWOOD:—As no one else seems inclined to discuss the paper, I should like to move that the Institute present to Mr. Kennelly a very hearty and cordial vote of thanks, for the intellectual treat which he has provided for us.

MR. SPRAGUE:—I should like to second the motion of Mr. Lockwood. I am prevented by an impedance in my own head from taking any part in the discussion. But I want to thank Mr. Kennelly for placing on the record, where it can be easily obtained, so much solid information, so clearly expressed, about a subject that is so little understood. While great knowledge is always of value if carried in one's own head; if it is placed where one can get at it with little trouble, and so that when gotten it can be easily digested, there is perhaps more benefit derived by the general run of electrical engineers than if they attempt to acquire too much personally. So I heartily second the motion that is made that a vote of thanks be tendered to Mr. Kennelly for his most interesting and valuable paper.

[The motion was carried.]

Adjourned.

[COMMUNICATED AFTER ADJOURNMENT BY MR. CHARLES P. STEINMETZ.]

I have read Mr. Kennelly's valuable paper on "Impedance" with the greatest interest, and am only sorry that I was not able to be present at its presentation.

One statement, however, I would like to emphasize somewhat more strongly, since in the paper it is enclosed between so many remarks of the highest practical value as to be liable to escape due notice, viz.: that

"Any circuit whatever, consisting of a combination of resistances, non-ferric inductances and capacities, carrying harmonically alternating currents, may be treated by the rules of unvarying currents, if the impedances are expressed by complex numbers:

$$a + bj = r(\cos\varphi + j\sin\varphi),$$

the algebraical operations being then performed according to the laws controlling complex quantities."

It is well known that the points of a plane can be represented by complex quantities in their rectangular representation

$$a + bj(^*),$$

or their polar representation

$$r(\cos\varphi + j\sin\varphi),$$

(*) Where j denotes $\sqrt{-1}$

and use has been made hereof repeatedly in the mathematical treatment of vector quantities. It is, however, the first instance here, so far as I know, that attention is drawn by Mr. Kennelly to the correspondence between the electrical term "impedance" and the complex numbers.

The importance hereof lies in the following:—The analysis of the complex plane is very well worked out, hence by reducing the electrical problems to the analysis of complex quantities they are brought within the scope of a known and well understood science.

Let us consider an instance hereof which will bring us to another point I intend to dwell upon.

In the alternating current line mentioned on page 187,[14] of Mr. Kennelly's paper, the impedance of the loop can be expressed by the complex quantity,

$$I_1 = 2.57 + 2.87j = 3.854(\cos 48° + j \sin 48°),$$

$$\text{or, } \frac{L}{2} = 1.285 + 1.135j$$
$$= 1.927(\cos 48° + j \sin 48°) \text{ per wire,}$$

that means, at the line current $C = 10$ amperes, between the ends of each line a difference of potential will exist of

$$\frac{CI_1}{2} = 19.27 \text{ volts.}$$

Let, now, the impedance of the generator be

$$I_0 = 1.5 + 10j = 10.11(\cos 81° + j \sin 81°),$$

then, if the consumer's circuit has an impedance

$$I_2 = 25.0 + 27.8j = 37.4(\cos 48° + j \sin 48°)$$

(containing for instance, motors, etc.) then the total impedance of the circuit is

$$I = I_0 + I_1 + I_2$$
$$= 29.07 + 40.67j = 50(\cos 54.5° + j \sin 54.5°)$$

In this case a generator potential of $E = 500$ volts will send a current of

$$C = \frac{E}{I} = 10 \text{ amperes}$$

through the circuit.

The difference of potential at the receiver end of the line will be

$$E_1 = CI_2 = 374 \text{ volts.}$$

At the generator end of the line

$$E_0 = C(I_1 + I_2) = 412.5 \text{ volts,}$$

and the drop of potential in the line,

$$E_0 - E_1 = 412.54 - 374 = 38.54 \text{ volts,}$$

or 19.27 volts per line, $= CI_1/2$, i.e., equal to the difference of potential between the ends of the line, or the true ohmic drop, times the impedance factor.

If, however, the receiving circuit has an impedance

$$I_2 = 442$$

(incandescent lamp load), the total impedance of the circuit is

$$I = 48.27 + 12.87j = 50(\cos 15° + j \sin 15°).$$

At the generator potential of 500 volts, 10 amperes will flow through the circuit again, but the difference of potential at the receiver end of the line is

$$E_1 = CI_2 = 44.2 \text{ volts.}$$

At the generator end,

$$E_0 = C(I_1 + I_2) = 468.6 \text{ volts,}$$

since

$$I_1 + I_2 = 46.77 + 2.87j = 46.86(\cos 4° + j \sin 4°)$$

so that the drop of potential in the line is $468.6 - 442 = 26.6$ ohms, or 13.3 volts per line. That is considerably less than the difference of potential between ends of the line, which still is 19.27 volts, so that in this case of the non-inductive, or lamp load, in spite of the impedance factor 1.5 of the line, the drop of potential of the line is practically only the true resistance drop, and the inductance of the line does not count at all.

Still more remarkable are the conditions if the receiver circuit has an impedance of a form similar to

$$I_2 = 15 - 60j = 61.95(\cos 76° - j \sin 76°),$$

as is the case if the line is feeding into a condenser of the capacity $K = 19$ mf. or running a synchronous motor (which under certain conditions of load and excitation acts like a condenser of large capacity).

In this case the total impedance of the line is

$$I = 19.07 - 46.22j = 50(\cos 67.7° - j \sin 67.7°).$$

Here again 500 volts generator E.M.F. will establish 10 amperes of current in the circuit.

But the difference of potential at the receiver end of the line is

$$E_1 = CI_2 = 619.5 \text{ volts.}$$

At the generator end,

$$E_0 = C(I_1 + I_2) = 598.7 \text{ volts}$$

that is; the drop of potential in the line will be

$$E_0 - E_1 = -20.80 \text{ volts,}$$
$$\text{or } - 10.4 \text{ volts per line;}$$

that means that, while a continuous current will experience a loss of potential of 12.85 volts per line wire, and from the impedance of the line a difference of potential of 19.27 volts per line wire is calculated, we find in this particular case a *rise* of potential of 10.4 volts per line wire, so that the difference of potential is *rising* along the line instead of *decreasing*, and that by not less than 20.8 volts.

In this case at an E.M.F. of 500 volts produced in the generator, an E.M.F. of 598.7 volts will appear at the generator terminals, and 619.5 at the receiver terminals.[15]

The conclusion that we derive herefrom is, that it is not permissible to calculate the drop of potential in an alternating circuit by multiplying the resistance drop by the impedance factor. The difference of potential between the ends of the line will indeed be equal to the current times impedance. The losses of potential in the line, however, will usually be less than the difference of potential between the ends of the line,

[14] TRANSACTIONS, vol. x, 1893. This issue, p. 466.

[15] It needs not to be remarked that if the generator contains iron, due to the variable permeability of iron the numerical values will, in practice, be found more or less modified.

and may even be negative; that is, the potential may rise in the line.

This is due to the difference of phase between the line impedance and the impedances of the other part of the circuit, as explained in the paper. I considered it advisable, however, to dwell upon this more particularly, since the mistake is made frequently, to calculate the drop of potential in an alternating current line from resistance, impedance factor and current. In reality the *drop of potential in one part of an alternating circuit, depends not only upon the constants of this part of the circuit, but upon the conditions of the other parts of the circuit also*, and may, therefore, at the same current strength vary enormously with different conditions of load.

Now a few words more on a question of terminology. Perusing the literature of the last years on this subject, we find everywhere the endeavor to establish a suitable name for quantities like

$$2\pi nL \text{ or } \frac{1}{2\pi nK}.$$

It is of the dimension "speed" or "resistance," is expressed in ohms, and defined by

$$\frac{\text{E.M.F. at right angles to current}}{\text{current}}.$$

This quantity and the true or ohmic resistance are the catheti of a rectangle with the impedance as hypotenuse.

Kennelly names it "*inductance-speed*" (the factor $2\pi n$ being an angular velocity).

This name, however, does not apply well to

$$\frac{1}{2\pi nK}.$$

About two years ago I proposed the name "*inductance*" for this quantity, so that impedance should be the resultant of resistance and inductance, as components. Unfortunately the name "*inductance*" has now been applied to what we called before the "*coefficient of self-induction*." S. P. Thompson in the new edition of his book has adopted the name "*inductance*" for $2\pi nL$ also.

The name "*inductive resistance*" has been used by Fleming and others. This name, however, has been applied also as synonymous with "*impedance*." "*Ohmic inductance*" has been proposed, but all these double names are too inconvenient to be of much practical value. Decidedly the best name would be "*inductance*," the more so as the constant L is so little used in practice that the ponderous name "*coefficient of self-induction*" will hardly cause any inconvenience.

Lynn, Mass., April, 1893.

[Reply to Mr. Steinmetz, communicated by the Author.]

Concerning Mr. Steinmetz's remarks, one in particular is of such practical importance that it will not probably suffer by emphasis or repetition. Mr. Steinmetz points out that when a circuit is composed of an alternator, supply conductor, and load, each having its individual impedance, their vector sum will generally be less than their arithmetical sum. Consequently if the generator delivers 1,100 volts at its terminals, and the drop in the supply wires allowing for the impedance factor is 100 volts (50 on each conductor, as might be indicated by a voltmeter connected between the far and near ends of one wire), the voltage at the transformer terminals would be never less, but usually more than 1,000 volts, particularly if condensers are in circuit at the receiving end, and its exact value would depend upon all the impedances in the circuit. It is therefore essential to observe that all the tabular impedance factors given in the paper as applying to sinusoidal currents, or augmented impedance formulas applying to non-sinusoidal currents, accurately yield the drop in the conductors so long as the static capacity in those conductors is negligible. The voltage of delivery at the distant end will always be greater than the difference between this drop and the voltage of supply at the near end, unless it happens that the impedances of the load and of the conductor have the same vector, or time constant. But unless condensers are added to the receiving end of the line, this variation of the actual voltage of delivery from the difference between line drop and voltage of supply, is generally small at full load on practical alternating transformer circuits. In other words the time constant of the ordinary supply wires does not differ materially from the apparent time constant of the transformers when loaded, although obviously this statement cannot be depended upon too far.