The scattering matrix

Shouri Chatterjee

September 21, 2014

As a student of circuit theory, I have always found the scattering matrix to be shrouded in mystery. Reading books, unfortunately, did not help me. This article is a ground-up attempt to uncover the mysteries behind the scattering matrix, and to provide a physical insight into its importance.

Consider the two-port network shown below. The two ports, 1 and 2, have voltages of V_1 and V_2, and currents of I_1 and I_2.

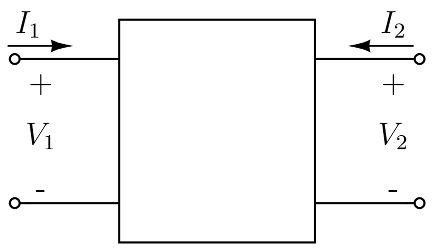

Figure 1: Two port network

1 Voltage and current waves

In the spirit of transmission-line analysis, let us hypothesize that any voltage can be declared to be a sum of a forward moving voltage wave, V^+, and a backward moving voltage wave, V^-. At the same time, the current will be the sum of a forward moving current wave, V^+/Z_0, and a backward moving current wave, $-V^-/Z_0$.

For the n-th port of an n-port network, V_n and I_n can therefore be written as:

$$\begin{aligned} V_n &= V_n^+ + V_n^- \\ I_n &= (V_n^+ - V_n^-)/Z_0 \end{aligned} \quad (1)$$

If we know V_n and I_n, then given the Z_0, it is possible to compute unique values for V_n^+ and V_n^-. (A pair of linear simultaneous equations, two unknowns.)

For the two-port network shown above, the forward and backward moving voltage waves at the two ports are related to each other through the scattering matrix.

Definition 1 (The scattering matrix)**.** The scattering matrix is defined as the relationship between the forward and backward moving waves. For a two-port network, like any other set of two-port parameters, the scattering matrix is a 2×2 matrix.

$$\begin{bmatrix} V_1^- \\ V_2^- \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} V_1^+ \\ V_2^+ \end{bmatrix} \tag{2}$$

It is instructive to point out here that the scattering matrix for a two-port network is always for a specified characteristic impedance, Z_0. In many cases, the characteristic impedance, Z_0 for different ports could be different. For example, if port-1 has a characteristic impedance of Z_{0_1}, and port-2 has a characteristic impedance of Z_{0_2}, then all that changes is the pair of simultaneous equations (1), for the two individual ports.

Corollary 1. The individual S-parameters can be computed from the definition of the scattering matrix using the following relationship:

$$S_{ij} = \left. \frac{V_i^-}{V_j^+} \right|_{V_k^+=0 \ \forall k \neq j} \tag{3}$$

2 Termination

Let us consider port-1 of the two-port network. To compute a parameter such as S_{22}, V_1^+ has to be forced to be 0. However, from (1), if V_1^+ is 0, the voltage at port-1 is just V_1^-, and the current is $-V_1^-/Z_0$. This means that port-1 is terminated with a resistor of value Z_0.

To force V_k^+ to 0, the k-th port has to be terminated with the corresponding characteristic impedance. (In most cases the characteristic impedance is specified to be the same for all ports. However, in general, each port is allowed to have its own characteristic impedance.)

Corollary 2. The computation of individual S-parameters in (3) can be re-written as:

$$S_{ij} = \left.\frac{V_i^-}{V_j^+}\right|_{\text{All other ports terminated by } Z_0} \tag{4}$$

3 Impedance looking into a port

Let us now consider the driving-point impedance looking into a port, say port-1. If we apply a voltage, V_1, and measure a current, I_1, then the driving-point impedance looking into port-1 is given by $Z_{in} = V_1/I_1$.

However, from (1), V_1 and I_1 are related to the forward and backward moving voltage waves. Z_{in} is therefore given by:

$$Z_{in} = Z_0 \cdot \frac{V_1^+ + V_1^-}{V_1^+ - V_1^-} \tag{5}$$

If port-2 (and all other ports for an n-port network) is terminated by its characteristic impedance, then V_1^- would be given by $S_{11}V_1^+$.

Corollary 3. The driving point impedance looking into port-1, with all other ports terminated with their characteristic impedances, is given by:

$$Z_{in} = Z_0 \cdot \frac{1 + S_{11}}{1 - S_{11}} \tag{6}$$

A componendo-dividendo operation on (6) will enable us to compute S_{11} from Z_{in}.

$$S_{11} = \frac{Z_{in} - Z_0}{Z_{in} + Z_0} \tag{7}$$

where Z_{in} is the driving-point impedance looking into port-1, and Z_0 is the characteristic impedance for port-1. The same result is applicable for any other port in an n-port network.

The results in (6) and (7) are similar to formulations for transmission lines, where we obtain the impedance looking into a transmission line when the reflection coefficient is Γ, or where we obtain the reflection coefficient at the load-end of the transmission line. After all, S_{11} is a reflection coefficient.

S_{11} is the reflection coefficient looking into port-1, when the other ports are terminated with their appropriate characteristic impedances.

4 Transmission

Let us now consider the voltage at port-2, when we apply a voltage at port-1. Without any loss of generality, let us consider a two-port network. Futher, let us assume that port-2 is terminated with its characteristic impedance, i.e, V_2^+ is 0.

Termination at port-2 implies that V_2 is just V_2^-. V_2^-/V_1^+ when port-2 is terminated with Z_0 is the definition for S_{21}. Therefore, if we apply a forward moving voltage wave V_1^+ at port-1, the voltage at port-2 is $S_{21}V_1^+$. However, there is also a reflected wave V_1^- at port-1, given by $S_{11}V_1^+$. Therefore, the voltage gain, V_2/V_1 is given by:

$$\frac{V_2}{V_1} = \frac{V_2^-}{V_1^+ + V_1^-} = \frac{V_2^-}{V_1^+(1+S_{11})} = \frac{S_{21}}{1+S_{11}} \tag{8}$$

If instead, we consider the current gain, I_2/I_1 is given by:

$$\frac{I_2}{I_1} = \frac{-V_2^-}{V_1^+ - V_1^-} \cdot \frac{Z_{0_1}}{Z_{0_2}} = \frac{-S_{21}}{1-S_{11}} \cdot \frac{Z_{0_1}}{Z_{0_2}} \tag{9}$$

The voltage gain in (8) could be used to compute the value of S_{21}. If we terminate port-2 with the appropriate characteristic impedance, apply a voltage V_1 at port-1, and measure a voltage V_2 at port-2, then S_{21} is given by:

$$S_{21} = \frac{V_2}{V_1}(1+S_{11}) \tag{10}$$

For circuits which have a driving-point impedance for port-1 exactly equal to the corresponding Z_0, i.e, $S_{11} = 0$, the voltage gain is equal to S_{21}.

5 Voltage and current gain

Let us now look at an experiment where the two-port network is being excited by a voltage source on port-1, with a source-impedance of Z_{0_1}. Further, the load on port-2 is exactly equal to Z_{0_2}.

The input impedance of the two-port network is given by (6). This gives us an input current of:

$$I_{in} = I_1 = \frac{V_{in}}{Z_{in} + Z_{0_1}} = V_{in}\frac{1-S_{11}}{2Z_{0_1}} \tag{11}$$

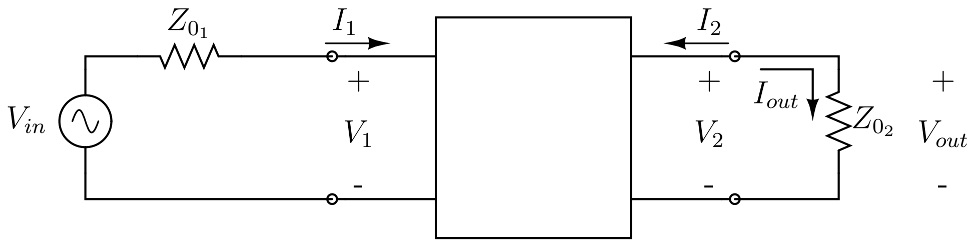

Figure 2: The two-port network being excited by a source of source resistance Z_{0_1}, and loaded with a load resistance Z_{0_2}.

Further, we can compute the voltage at port-1:

$$V_1 = I_1 Z_{in} = I_{in} Z_{in} = V_{in} \frac{1 + S_{11}}{2} \tag{12}$$

(8) and (12) can be combined to obtain the overall voltage gain from the input to the output.

$$\frac{V_{out}}{V_{in}} = \frac{V_2}{V_{in}} = \frac{V_2}{V_1} \cdot \frac{V_1}{V_{in}} = \frac{S_{21}}{2} \tag{13}$$

Corollary 4. When a two-port network is excited on port-1 with a voltage source of source-impedance Z_{0_1}, and when there is a load of Z_{0_2} on port-2, the voltage gain from the input to the output is directly given by $S_{21}/2$.

The current gain can also be computed similarly. I_{out} is given by $-I_2$; I_2/I_1 is given by (9).

$$\frac{I_{out}}{I_{in}} = -\frac{I_2}{I_1} = \frac{S_{21}}{1 - S_{11}} \cdot \frac{Z_{0_1}}{Z_{0_2}} \tag{14}$$

6 Power

With the help of the scattering matrix, we would like to compute the power delivered to the two-port network, and the power delivered to the load, Z_{0_2}.

With the help of (13), the power delivered to the load is given by:[1]

$$P_{out} = V_{out} I_{out}^* = V_{in} \frac{S_{21}}{2} \cdot V_{in}^* \frac{S_{21}^*}{2Z_{0_2}} = \frac{|V_{in}|^2}{4Z_{0_2}} |S_{21}|^2 \tag{15}$$

$|V_{in}|^2/4Z_{0_1}$ is the maximum available power from the voltage source V_{in}. (Recall the maximum power transfer theorem. For completeness, the maximum power transfer theorem is being discussed in Appendix B.)

Corollary 5. The power delivered to the load, through the two-port network, is given by the maximum available power from the voltage source, P_{max}, times $|S_{21}|^2$, times a ratio of the characteristic impedance of the two ports. That is:

$$P_{out} = P_{max} |S_{21}|^2 \frac{Z_{0_1}}{Z_{0_2}} \tag{16}$$

Corollary 6. When all the ports have equal characteristic impedance, the power delivered to the load is simply given by:

$$P_{out} = P_{max} |S_{21}|^2 \tag{17}$$

The power delivered to the two-port network, into port-1, is given with the help of (12) and (11), by:

$$P_1 = V_1 I_1^* = V_{in} \frac{1 + S_{11}}{2} \cdot V_{in}^* \frac{1 - S_{11}^*}{2Z_{0_1}} = \frac{|V_{in}|^2}{4Z_{0_1}} (1 + 2j\,\mathrm{Im}(S_{11}) - |S_{11}|^2) \tag{18}$$

P_1 includes both real power and reactive power. The real (active) and reactive powers delivered to the two-port network are given by:

$$\mathrm{Re}(P_1) = \frac{|V_{in}|^2}{4Z_{0_1}} (1 - |S_{11}|^2) = P_{max}(1 - |S_{11}|^2) \tag{19}$$

$$\mathrm{Im}(P_1) = \frac{|V_{in}|^2}{4Z_{0_1}} \cdot 2\,\mathrm{Im}(S_{11}) = P_{max} \cdot 2\,\mathrm{Im}(S_{11}) \tag{20}$$

[1]Revelation! All through we have been assuming that all the voltages and currents are phasors. However, we are also assuming that Z_{0_1} and Z_{0_2} are real quantities. After all, these are the "characteristic" impedances of the two ports, and can be chosen arbitrarily. These are also the source and load resistances respectively, in our experiment of Fig. 2. Power in phasors is discussed in Appendix A.

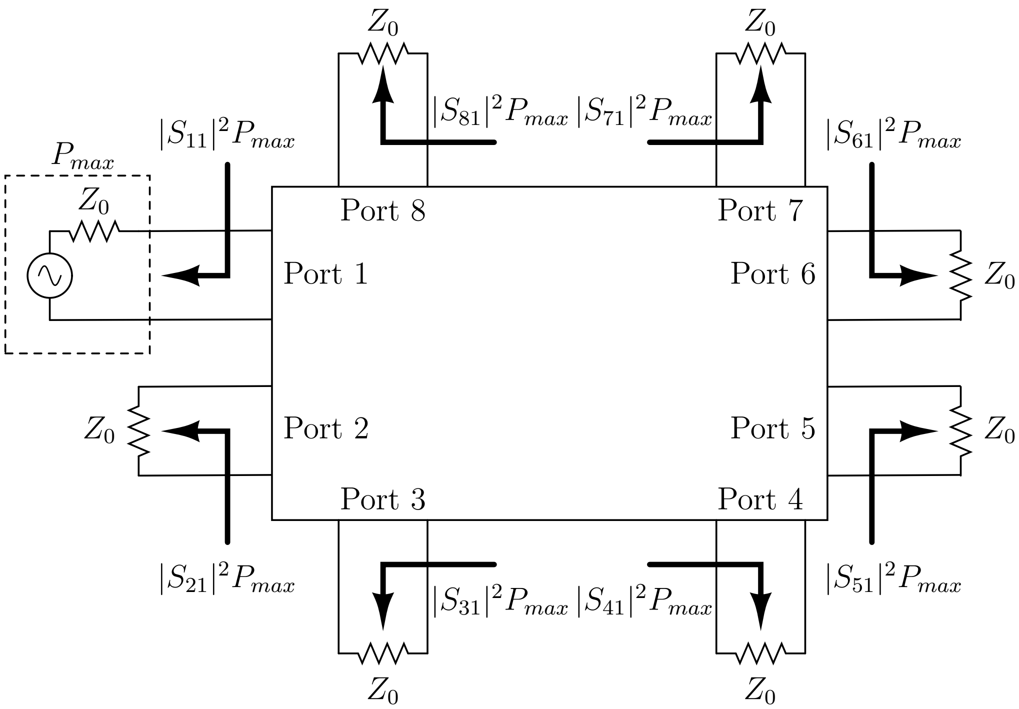

Figure 3: Power distribution in an 8-port network. All ports have a characteristic impedance of Z_0.

(19) is significant. It tells us that the real power delivered to the two port network is $(1 - |S_{11}|^2)$ times the maximum available power from the source. If S_{11} is 0, we get maximum power transfer, and P_{max} is delivered to the two-port network. Otherwise, $P_{max}|S_{11}|^2$ is the power reflected back to the source. Further, to recapitulate, $P_{max}|S_{21}|^2\frac{Z_{0_1}}{Z_{0_2}}$ is the power delivered to the load.

Fig. 3 shows an 8-port network, where all the ports have the same characteristic impedance, Z_0. The scattering matrix of this 8-port network is an 8×8 matrix. All the ports are terminated in Z_0. Now, let us consider an experiment where the 8-port network is excited on port-1 with a source of source impedance Z_0, and maximum available power of P_{max}. The scattering matrix tells us how power is distributed to the 8 ports. That is, a power of $|S_{11}|^2P_{max}$ is reflected back to the source, a power of $|S_{21}|^2P_{max}$ is delivered to port-2, a power of $|S_{31}|^2P_{max}$ is delivered to port-3, etc., as shown in the figure.

7 Lossless passive two-port networks

Let us further consider a passive two-port network which has only ideal inductors, capacitors and mutual inductances, with any arbitrary configuration within it. The components of the two-port network themselves will not consume any real (active) power.

In such a case, the real (active) power delivered to the two-port network into port-1 will be completely delivered to the load on port-2. From (19) and (15), this means:

$$P_{max}(1 - |S_{11}|^2) = P_{max}|S_{21}|^2 \cdot \frac{Z_{0_1}}{Z_{0_2}}$$

Corollary 7. For a lossless two-port network, therefore, from the above:

$$|S_{11}|^2 + |S_{21}|^2 \cdot \frac{Z_{0_1}}{Z_{0_2}} = 1 \tag{21}$$

Corollary 8. When the two ports of a lossless two-port network have a uniform characteristic impedance,

$$|S_{11}|^2 + |S_{21}|^2 = 1 \tag{22}$$

8 Reciprocity

The reciprocity theorem tells us the following for a passive two-port network:

- We apply a voltage, V, at port-1 and measure the short-circuit current at port-2 to be I.
- If we apply the same voltage, V, at port-2, we will find the short-circuit current at port-1 to be I.

If a two-port network is reciprocal (i.e., passive), the two-port network shown in Fig. 4 is also reciprocal.

If we apply a voltage V at port-1 of the bigger two-port network, and short circuit port-2 of the bigger two-port network, then the voltage at port-2 of

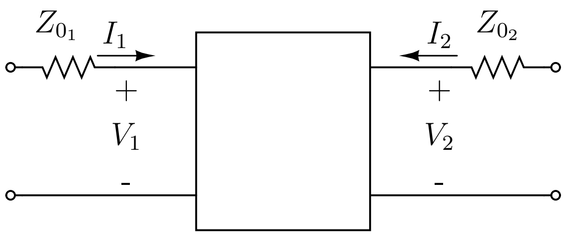

Figure 4: Larger two-port network, including the terminations

the smaller two-port network will be given, with the help of (13), as:

$$V_2 = S_{21}\frac{V}{2}$$

This gives a short-circuit current at port-2 as:

$$-I_2 = S_{21}\frac{V}{2Z_{0_2}}$$

Conversely, if we apply the same voltage at port-2 of the bigger two-port network and measure the short circuit current at port-1 of the bigger two-port network, the short circuit current will be given by:

$$-I_1 = S_{12}\frac{V}{2Z_{0_1}}$$

By the reciprocity theorem, the short circuit currents for the two experiments are equal.

Corollary 9. For a reciprocal network:

$$\frac{S_{21}}{Z_{0_2}} = \frac{S_{12}}{Z_{0_1}} \tag{23}$$

If the characteristic impedances for the two ports are equal, $S_{21} = S_{12}$.

Appendices

A Power computations with phasors

Let us compute the power consumed by a device, where the current through the device is given by $i(t) = \sqrt{2} \cdot I \cos(\omega t + \phi_1)$, and the voltage across the device is $v(t) = \sqrt{2} \cdot V \cos(\omega t + \phi_2)$. The phasor corresponding to $i(t)$ is $I\angle\phi_1$, or, $Ie^{j\phi_1}$, or, $I\cos\phi_1 + jI\sin\phi_1$. The phasor corresponding to $v(t)$ is $V\angle\phi_2$, or, $Ve^{j\phi_2}$, or, $V\cos\phi_2 + jV\sin\phi_2$.

The average power consumed over one period is given by:

$$\begin{aligned}
P &= \frac{\omega}{2\pi}\int_0^{2\pi/\omega} v(t)i(t)dt = \frac{\omega}{\pi}\cdot VI \cdot \int_0^{2\pi/\omega} \cos(\omega t + \phi_1)\cos(\omega t + \phi_2)dt \\
&= VI\frac{\omega}{2\pi}\int_0^{2\pi/\omega} (\cos(2\omega t + \phi_1 + \phi_2) + \cos(\phi_2 - \phi_1))dt \\
&= VI\frac{\omega}{2\pi}\int_0^{2\pi/\omega} \cos(\phi_2 - \phi_1)dt \\
&= VI\cos(\phi_2 - \phi_1) \qquad (24)
\end{aligned}$$

This also happens to be the real part of the product of the phasor corresponding to $v(t)$ and the conjugate of the phasor corresponding to $i(t)$.

$$P = VI\cos(\phi_2 - \phi_1) = \mathrm{Re}(Ve^{j\phi_2} \cdot Ie^{-j\phi_1}) = \mathrm{Re}(\boldsymbol{V} \cdot \boldsymbol{I}^*) \qquad (25)$$

$P = \mathrm{Re}(\boldsymbol{V} \cdot \boldsymbol{I}^*)$, is also known as the "active power" consumed by the device. The imaginary part, $P_r = \mathrm{Im}(\boldsymbol{V} \cdot \boldsymbol{I}^*)$ is called the "reactive power" consumed by the device. If the device is split into a real resistance and a reactance, the resistance will consume the active power. The reactance will store energy. The rate of change of stored energy in the reactance will have an amplitude equal to the reactive power. The total reactive power in a circuit will be equal to 0, and can be proved with the help of Tellegen's theorem.

B Maximum power transfer theorem

The maximum power transfer theorem is often misunderstood. This appendix is to prove and clarify what the maximum power transfer theorem is all about.

Fig. 5 shows a voltage source, of source impedance Z_S, trying to deliver power to a load of load impedance Z_L. **The basic premise of the rest of the discussion is that the source impedance is fixed.** No matter what you do, you are not allowed to change the source impedance.

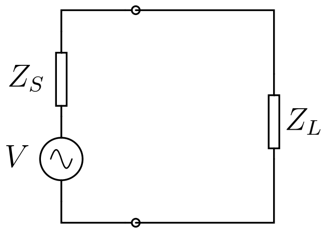

Figure 5: Source trying to deliver power to a load

Theorem 1. A source with source impedance Z_S is able to deliver maximum power to a load when the load impedance, Z_L, is equal to Z_S^*.

Proof. The current through the circuit is $V/(Z_S + Z_L)$. The voltage across Z_L is $VZ_L/(Z_S + Z_L)$. Therefore the power delivered to Z_L is:

$$P_L = V\frac{Z_L}{Z_S + Z_L} \cdot \frac{V^*}{Z_S^* + Z_L^*} = |V|^2\frac{Z_L}{|Z_S + Z_L|^2}$$

If Z_S is $R_S + jX_S$, and if Z_L is $R_L + jX_L$, the power delivered to the load Z_L will become:

$$P_L = |V|^2\frac{R_L + jX_L}{(R_S + R_L)^2 + (X_S + X_L)^2}$$

The real (active) part of this power is:

$$\mathrm{Re}(P_L) = |V|^2\frac{R_L}{(R_S + R_L)^2 + (X_S + X_L)^2} \tag{26}$$

If we need maximum power delivered to the Z_L, the derivatives of the (26) with respect to R_L and X_L will have to be 0. The derivative with respect to R_L being 0 implies:

$$\begin{aligned}\frac{d}{dR_L}\left(\frac{R_L}{(R_S + R_L)^2 + (X_S + X_L)^2}\right) &= 0 \\ \Rightarrow\ 2R_L(R_S + R_L) - (R_S + R_L)^2 &= 0 \\ \Rightarrow\ R_L &= R_S \qquad (27)\end{aligned}$$

Further, the derivative with respect to X_L being 0 implies:

$$\begin{aligned}\frac{d}{dX_L}\left(\frac{R_L}{(R_S + R_L)^2 + (X_S + X_L)^2}\right) &= 0 \\ \Rightarrow\ 2R_L(X_L + X_S) &= 0 \\ \Rightarrow\ X_L &= -X_S \qquad (28)\end{aligned}$$

(27) and (28) indicate that $Z_L = R_S - jX_S$, or in other words, Z_L has to be Z_S^*. □

The maximum power that can be transferred from a source with a real source resistance of R_S, is given by the power delivered to a load of value R_S. If the voltage of the source is V, the maximum power that it can deliver is therefore:

$$P_{max} = \frac{|V|^2}{4R_S} \tag{29}$$

In many situations, a source of power is defined by the maximum power it can deliver. Instead of specifying its voltage (or current) and source impedance, often one can specify the maximum power that it can deliver, or the "maximum available power".