Lecture 1

1 Real number system

We are familiar with natural numbers and to some extent the rational numbers. While finding roots of algebraic equations we see that rational numbers are not enough to represent roots which are not rational numbers. For example draw the graph of $y = x^2 - 2$. We see that it cross the x-axis twice. The roots are such that their square is 2, but they cannot be rational numbers according to the following theorem.

Theorem 1.0.1. *Suppose that $a_0, a_1, ..., a_n (n \geq 1)$ are integers such that $a_0 \neq 0, a_n \neq 0$ and that r satisfies the equation*

$$a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0 = 0.$$

If $r = \frac{p}{q}$ where p, q are integers with no common factors and $q \neq 0$. Then q divides a_n and p divides a_0.

Proof: Since $\frac{p}{q}$ satisfies the equation, we have

$$a_n p^n + a_{n-1} p^{n-1} q + ... + a_0 q^n = 0$$

i.e., $a_n p^n = -q(a_{n-1} p^{n-1} + ... + a_0 q^{n-1})$. This means q divides a_n as p, q have no common factors. On the other hand we can also write

$$a_0 q^n = -p(a_n p^{n-1} + a_{n-1} p^{n-2} + ... + a_1 q^{n-1}).$$

Thus p divides a_0. (Here we used the following: p, q can be expressed as $p = p_1 ... p_j$ and $q = q_1 ... q_k$ where each p_i, q_i are prime numbers. Since p divides $a_0 q^n$, the quantity

$$\frac{a_0 q^n}{p} = a_0 \frac{q_1^n ... q_k^n}{p_1 ... p_j}$$

must be an integer. Since no p_i is equal to q_j, the prime factorization of a_0 must include the product $p_1 ... p_j$.) ///

Now we see that the possible rational roots of $x^2 - 2 = 0$ are $\pm 1, \pm 2$. But it is easy to check that $\pm 1, \pm 2$ does not satisfy $x^2 - 2 = 0$. So the roots of $x^2 - 2 = 0$ are not rational numbers. This means the set of rational numbers has "gaps". So the natural question to ask is: Can we have a number system without these gaps? The answer is yes and the "complete number system" with out these gaps is the real line $\mathbb{R}$. We will not look into the development of $\mathbb{R}$ as it is not easy to define the real numbers. We assume that there is a set $\mathbb{R}$, whose elements are called real numbers and $\mathbb{R}$ is closed with respect to addition and multiplication. That is, given any $a, b \in \mathbb{R}$, the sum $a + b$ and product ab also represent real numbers. Moreover, $\mathbb{R}$ has an order structure $\leq$ and has no "gaps" in the sense that it satisfies the Completeness Axiom(see below).

Let S be a non-empty subset of $\mathbb{R}$. If S contains a largest element s^0, then we call s^0 the maximum of S. If S contains a smallest element s_0, then we call s_0 the minimum of S. If S is bounded above and S has least upper bound, then we call it the supremum of S. If S is bounded below and S has greatest lower bound, then we call it as infimum of S.

Unlike maximum and minimum, $\sup S$ and $\inf S$ need not belong to the set S. An important observation is if $\alpha = \sup S$ is finite, then for every $\epsilon > 0$, there exists an element $s \in S$ such that $s \geq \alpha - \epsilon$.
Note that any bounded subset of Natural numbers has maximum and minimum.

Completeness Axiom: Every nonempty subset S of $\mathbb{R}$ that is bounded above has a least upper bound. In other words, $\sup S$ exists and is a real number.

The completeness axiom does not hold for $\mathbb{Q}$. That is, every non-empty subset of $\mathbb{Q}$ that is bounded above by a rational number need not have rational least upper bound. For example $\{r \in \mathbb{Q} : r^2 \leq 2\}$.

Archimedean property:

Theorem 1.0.2. *For each $x \in \mathbb{R}$, there exists a natural number $N = N(x)$ such that $x < N$.*

Proof: Assume by contradiction that this is not true. Then there is no $N \in \mathbb{N}$ such that $x < N$. i.e., x is an upper bound for $\mathbb{N}$. Then, by completeness axiom, let u be the smallest such bound of $\mathbb{N}$ in $\mathbb{R}$. That is $u \in \mathbb{R}$ and so $u - m$ for $2 \leq m \in \mathbb{N}$ is not an upper bound for $\mathbb{N}$. Therefore, there exists $k \in \mathbb{N}$ such that $u - m < k$, but then $u < k + m$, and $k + m \in \mathbb{N}$. a contradiction. ///

Now it is easy to see the following corollary

Corollary 1.0.3. *Let $S = \{\frac{1}{n} : n \in \mathbb{N}\}$. Then $w = \inf S = 0$.*

Proof: We note that S is bounded below. Let $\epsilon > 0$ be an arbitrary positive real number. By above Archimedean property, there exists $n \in \mathbb{N}$ such that $n > \frac{1}{\epsilon}$. Then we have,

$$0 \leq w \leq \frac{1}{n} < \epsilon.$$

Since ϵ is arbitrary, we have $w = 0$. (why?)

Corollary 1.0.4. *If $y > 0$ be a real number, then there exists $n = n(y) \in \mathbb{N}$ such that $n - 1 \leq y < n$.*

Finally, we have the following density theorem

Theorem 1.0.5. *Let x, y are real numbers such that $x < y$. Then there exists a rational number q such that $x < q < y$.*

Proof: Without loss of generality, assume that $x > 0$. Now let $n \in \mathbb{N}$ be such that $y - x > \frac{1}{n}$ (Archimedean property). Now consider the set

$$S = \{m \in \mathbb{N} : \frac{m}{n} > x\}.$$

Then S is non-empty (by Archimedean property). By well -ordering of $\mathbb{N}$, S has minimal element say m_0. Then $x < \frac{m_0}{n}$. By the minimality of m_0, we see that $\frac{m_0 - 1}{n} \leq x$. Then,

$$\frac{m_0}{n} \leq x + \frac{1}{n} < x + (y - x) = y.$$

Therefore,

$$x < \frac{m_0}{n} < y.$$

Definition 1.0.6. *We say a subset E of $\mathbb{R}$ is countable if either E is finite or there is a bijection between $\mathbb{N}$ and E.*

We can define a map from $\mathbb{N}$ to Q following some order. On the otherhand it can be shown that countable union of finite sets is also countable(easy to see this).

Remark 1.1. *$\mathbb{Q}$ is countable. Indeed, for each $n \in \mathbb{N}$, define the set*

$$E_n = \{0 < r \in Q, r = \frac{p}{q}, p + q = n\}$$

For example $E_2 = \{\frac{1}{1}\}, E_3 = \{\frac{2}{1}, \frac{1}{2}\}, E_4 = \{\frac{1}{3}, \frac{2}{2}, \frac{3}{1}\}$. Each E_n contains finitely many elements and $Q^+ = \cup_n E_n$.