Lecture 10

1 Test for convergence ctd..

Theorem 1.0.1 (Ratio test)**.** *Let* $\sum_1^\infty a_n$ *be a series of real numbers. Let*

$$a = \liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \text{ and } A = \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

Then

a) $\sum_1^\infty a_n$ *converges absolutely if* $A < 1$;

b) $\sum_1^\infty |a_n|$ *diverges if* $a > 1$;

c) the test fails in all other cases.

Proof. a) If $A < 1$, *choose* B *such thata* $A < B < 1$. *Then there exists an* $\epsilon > 0$ *such that* $B = A + \epsilon$ *and also* $N \in \mathbb{N}$ *such that* $\left|\frac{a_{n+1}}{a_n}\right| \leq B$ *for all* $n \geq N$. *Further, for any* $k \in \mathbb{N}$,

$$\left|\frac{a_{N+k}}{a_N}\right| = \prod_{i=1}^{k} \left|\frac{a_{N+i}}{a_{N+i-1}}\right| \leq \prod_{i=1}^{k} B = B^k.$$

Thus $|a_{N+k}| \leq B^k |a_N|,\ k \in \mathbb{N}$. *But* $\sum_{k=0}^{\infty} |a_N| B^k < \infty$ *as* $B < 1$. *Thus by comparison test, the series* $\sum_1^\infty a_n$ *converges.*

b) If $a > 1$, *choose* b *such that* $1 < b < a$. *There exits* $N \in \mathbb{N}$ *such that* $\left|\frac{a_{n+1}}{a_n}\right| \geq b$ *for all* $n \geq N$. *Further, for any* $k \in \mathbb{N}$,

$$\left|\frac{a_{N+k}}{a_N}\right| = \prod_{i=1}^{k} \left|\frac{a_{N+i}}{a_{N+i-1}}\right| \geq \prod_{i=1}^{k} b = b^k.$$

Thus $|a_{N+k}| \geq |a_N|,\ k \in \mathbb{N}$. *But, as* $b > 1$, $\sum_{k=0}^{\infty} a_N b^k$ *diverges. Thus, again, by the comparison test, the series* $\sum_1^\infty a_n$ *diverges.*

c) Case1: $a = A = 1$ *Consider the series* $\sum \frac{1}{n}$. *Here* $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$. *But* $\sum \frac{1}{n}$ *diverges. For the series* $\sum \frac{1}{n^2}$, *which converges, again* $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$.
Case 2: $A > 1$ *If we consider the series* $\sum 2^n$ *then* $A = 2 > 1$ *and the series diverges. If we take*

$$s = 1 + 2 + \frac{1}{5} + \frac{2}{5} + (\frac{1}{5})^2 + 2(\frac{1}{5})^2 + (\frac{1}{5})^3 + ...$$

Then it is easy to see that the series converges as

$$s = 1 + (\frac{1}{5}) + (\frac{1}{5})^3 + ... + 2 + 2(\frac{1}{5}) + 2(\frac{1}{5})^2 + 2(\frac{1}{5})^3 + ...$$

But $A = 2$. Similarly one can construct examples when $a < 1$. ///

Examples 1.0.2.

a) *Consider the series $\sum_1^\infty \frac{n^n}{n!}$. Here*

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!}\frac{n!}{n^n} = \left(\frac{n+1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^n \to e,$$

which is greater than 1. So $a = A = e > 1$. Thus the given series diverges.

b) *Consider the series $\sum_0^\infty \frac{x^n}{n!}, x \in \mathbb{R}$. Here*

$$\frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)!}\frac{n!}{x^n} = \frac{x}{n+1} \to 0.$$

Therefore $a = A = 0 < 1$. Thus, for all $x \in \mathbb{R}$, the given series converges.

Theorem 1.0.3 (Root test). *Let $\sum_1^\infty a_n$ be a series of real numbers. Let $A = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. Then*

a) *the series converges absolutely if $A < 1$;*

b) *the series diverges if $A > 1$;*

c) *the test fails if $A = 1$.*

Proof. *a) If $A < 1$, choose B such that $A < B < 1$. Then there exists $N \in \mathbb{N}$ such that $\sqrt[n]{|a_n|} < B$ for all $n \geq N$. This implies $|a_n| < B^n$ for all $n \geq N$. As $B < 1$, the series converges by comparison test.*

b) If $A > 1$, there exists infinitely many $n \in \mathbb{N}$ such that $\sqrt[n]{|a_n|} > 1$. But this implies that $|a_n| > 1$ for infinitely many values of n and hence $a_N \nrightarrow 0$, i.e., $\sum a_n$ diverges.

c) Consider the series $\sum \frac{1}{n}$. Here $A = 1$ and the series diverges. On the other hand, for the series $\sum \frac{1}{n^2}$, again $A = 1$, but the series converges. ///

Examples 1.0.4.

1) *Consider the series $\sum_1^\infty \frac{x^n}{n}, x \in \mathbb{R}$. Here $a_n = \frac{x^n}{n}$. Therefore, $\sqrt[n]{\left|\frac{x^n}{n}\right|} = \left|\frac{x}{\sqrt[n]{n}}\right| \to |x|$. Thus the series converges for $|x| < 1$ and diverges for $|x| > 1$.*

2) *Consider the series $\sum_1^\infty \frac{x^n}{n^n}, x \in \mathbb{R}$. Here $a_n = \frac{x^n}{n^n}$. Then, $\sqrt[n]{|a_n|} = \left|\frac{x}{n}\right| \to 0$. Thus the series converges for any $x \in \mathbb{R}$.*

3) *Consider the series* $\sum a_n$, *where* $a_n = \begin{cases} \dfrac{n}{4^n} & n \text{ is odd} \\ \dfrac{1}{2^n} & n \text{ is even} \end{cases}$. *Then* $\limsup_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{2}$. *Therefore the series converges.*

4) *The series* $\sum 3^{-n-(-1)^n}$. *Then it is not difficult to see that* $\limsup |a_n|^{\frac{1}{n}} = 1/3$. *However ratio test fails in this case.*

Remark 1.1. *We note that the root test is stronger than the ratio test. for example, take the series* $\sum a_n$ *where*

$$a_n = \begin{cases} 2^{-n} & n \text{ odd} \\ 2^{-n+2} & n \text{ even} \end{cases}$$

Then it is easy to see that

$$\limsup \frac{|a_{n+1}|}{|a_n|} = 2, \text{but} \quad \limsup |a_n|^{1/n} = 1/2.$$

So the root test implies that the series converges but ratio test is inconclusive.