Lecture 11

1 Alternating series

Definition 1.0.1. *An alternating series is an infinite series whose terms alternate in sign.*

Theorem 1.0.2. *Suppose $\{a_n\}$ is a sequence of positive numbers such that*

(a) $a_n \geq a_{n+1}$ *for all* $n \in \mathbb{N}$ *and*

(b) $\lim\limits_{n\to\infty} a_n = 0$,

then the alternating series $\sum\limits_{n=1}^{\infty} (-1)^{n+1} a_n$ *converges.*

Proof. Consider the partial sums with odd index, $s_1, s_3, s_5, \ldots$. Now, for any $n \in \mathbb{N}$,

$$s_{2n+1} = s_{2n-1} - a_{2n} + a_{2n+1} \leq s_{2n-1} \text{ (by (a))}.$$

Thus the sequence $\{s_{2n-1}\}_1^{\infty}$ forms a non-increasing sequence. Also, notice that

$$s_{2n-1} = \sum_{i=1}^{n-1} (a_{2i-1} - a_{2i}) + a_{2n-1}.$$

Since each quantity in the parenthesis is non-negative and $a_{2n-1} > 0$, the sequence $\{s_{2n-1}\}$ is bounded below by 0. Hence $\{s_{2n-1}\}_1^{\infty}$ is convergent.

Now, consider the partial sums with even index, $s_2, s_4, s_6, \ldots$. For any $n \in \mathbb{N}$,

$$s_{2n+2} = s_{2n} + a_{2n+1} - a_{2n+2} \geq s_{2n} \text{ (by (a))}.$$

Thus the sequence $\{s_{2n}\}_1^{\infty}$ forms a non-decreasing sequence. Further,

$$s_{2n} = a_1 - \sum_{i=1}^{n-1} (a_{2i} - a_{2i+1}) - a_{2n} \leq a_1,$$

which means that s_{2n} is bounded above by a_1. Therefore, $\{s_{2n}\}$ is convergent.

Let $L = \lim\ s_{2n}$ and $M = \lim\ S_{2n-1}$. By ((b)),

$$0 = \lim\ a_{2n} = \lim\ (s_{2n} - s_{2n-1}) = L - M.$$

Thus $L = M$ and hence the alternating series $\sum\limits_{n=1}^{\infty} (-1)^{n+1} a_n$ converges. ///

Examples 1.0.3.

1) *Consider the series* $\sum\limits_{n=1}^{\infty} (-1)^{n+1} 2^{1/n}$. *Here* $a_n = 2^{1/n} \to 1$ *as* $n \to \infty$. *Hence the above theorem does not apply. Anyhow, one can show that the series diverges.*

2) Consider the series $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. *The* $a_n's$ *of this series satisfies the hypothesis of the above theorem and hence the series converges.*

Examples 1.0.4.

1) The series $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ *converges conditionally.*

2) The series $\sum\limits_{n=1}^{\infty} \frac{(-1)^{2n-1}}{2n-1}$ *converges conditionally.*

The following is a more genreal test than the previous theorem.

Theorem 1.0.5. *(Dirichlet test)*
Let $\{a_n\}$ *and* $\{b_n\}$ *be sequences of real numbers such that*

1. *the sequence* $s_n = \sum\limits_{k=1}^{n} a_k$ *is bounded,*

2. *the sequence* b_n *is decreasing and* $b_n \to 0$.

Then the series $\sum a_n b_n$ *converges.*

Proof. Let $t_n = \sum\limits_{k=1}^{n} a_k b_k$. Since s_n is bounded, there exists $M > 0$, such that $|s_n| \leq M$ for all n. Now note that

$$a_1 b_1 + a_2 b_2 + ... + a_n b_n = s_1(b_1 - b_2) + s_2(b_2 - b_3) + ...s_{n-1}(b_{n-1} - b_n) + s_n b_n$$

Since b_n is decreasing, $b_n - b_{n+1} \geq 0$. Therefore

$$|a_1 b_1 + a_2 b_2 + ... + a_n b_n| \leq M b_1$$

Since $b_n \to 0$, for any $\epsilon > 0$, we get N such that $|b_n| \leq \epsilon$ for all $n \geq N$. Now we can easily see that for $m > n$,

$$|t_m - t_n| = |\sum_{n}^{m} a_k b_k| = M|b_n| \leq M\epsilon$$

Therefore, by Cauchy's test, the series $\sum a_n b_n$ converges. ///

Examples 1.0.6.

1) Consider the series $\sum \frac{\cos n\pi}{\log n}$. *Here take* $a_n = \cos n\pi$ *and* $b_n = \frac{1}{\log n}$. *Then*

$$|A_n| \leq |\sum_{k=1}^{n} \cos n\pi| \leq 1$$

(check the first 4 terms and then use periodicity of $\cos x$*)*
and b_n *decreases to* 0. *Hence the series conveges. In this case we can see that the series does not converge absolutely (apply Cauchy's test).*

2) $\sum \frac{2^{2n}n^2}{e^n n!}\frac{1}{(\log n)^2}$. *Take* $b_n = \frac{1}{(\log n)^2}$ *and* $a_n = \frac{2^{2n}n^2}{e^n n!}$. *Then* b_n *decreases to* 0. *To show the boundedness of the partial sums of* $\sum a_n$*, we can apply Ratio test to see that the series* $\sum a_n$ *converges. Hence the sequence of partial sums converge and so will be bounded. Therefore by Dirichlet test the series* $\sum a_n b_n$ *converges.*

Theorem 1.0.7. *(Integral Test). If* $f(x)$ *is decreasing and non-negative on* $[1,\infty)$*, Then*

$$\int_1^\infty f(x)dx < \infty \iff \sum_{n=1}^{\infty} f(n) \text{ converges.}$$

Details of this theorem will be done after convergence of improper integral.

Examples 1.0.8. *1.* $f(x) = \frac{1}{x}$. *Here* $\int_0^\infty f(x)dx$ *is not finite. Therefore* $\sum \frac{1}{n}$ *diverges.*

2. $f(x) = \frac{1}{x^2}$. *Here* $\int_0^\infty f(x)dx$ *is finite. Therefore* $\sum \frac{1}{n^2}$ *converges.*