Lecture 12

1 Calculus of real valued functions

Let $f(x)$ be defined on (a, b) except possibly at x_0.

Definition 1.0.1. *We say that* $\lim_{x\to x_0} f(x) = L$ *if, for every real number* $\epsilon > 0$*, there exists a real number* $\delta > 0$ *such that*

$$0 < |x - x_0| < \delta \implies |f(x) - L| < \epsilon. \tag{1.1}$$

Equivalently,

Remark 1.1. *The above definition is equivalent to: for any sequence* $\{x_n\}$ *with* $x_n \to x_0$*, we have* $f(x_n) \to L$ *as* $n \to \infty$*.*

Proof. Suppose $lim_{x\to x_0} f(x)$ exists. Take $\epsilon > 0$ and let $\{x_n\}$ be a sequence converging to x_0. Then there exists N such that $|x_n - x_0| < \delta$ for $n \geq N$. Then by the definition $|f(x_n) - L| < \epsilon$. i.e., $f(x_n) \to L$.
For the other side, assume that $x_n \to c \implies f(x_n) \to L$. Suppose the limit does not exist. i.e., $\exists\ \epsilon_0 > 0$ such that for any $\delta > 0$, there is $x \in |x - x_0| < \delta$, for which $|f(x) - L| \geq \epsilon_0$. Then take $\delta = \frac{1}{n}$ and pick x_n in $|x_n - x_0| < \frac{1}{n}$, then $x_n \to x_0$ but $|f(x_n) - L| \geq \epsilon_0$. Not possible.

Theorem 1.0.2. *If limit exists, then it is unique.*

Proof. Proof is easy.

Examples 1.0.3. *1.* $\lim_{x\to 1}(\frac{3x}{2} - 1) = \frac{1}{2}$*. Let* $\epsilon > 0$*. We have to find* $\delta > 0$ *such that* (1.1) *holds with* $L = 1/2$*. Working backwards,*

$$\frac{3}{2}|x - 1| < \epsilon \textit{ when ever } |x - 1| < \delta := \frac{2}{3}\epsilon.$$

2. Prove that $\lim_{x\to 2} f(x) = 4$*, where* $f(x) = \begin{cases} x^2 & x \neq 2 \\ 1 & x = 2 \end{cases}$

Problem: Show that $\lim_{x\to 0} \sin(\frac{1}{x})$ does not exist.
Consider the sequences $\{x_n\} = \{\frac{1}{n\pi}\}, \{y_n\} = \{\frac{1}{2n\pi + \frac{\pi}{2}}\}$. Then it is easy to see that $x_n, y_n \to 0$ and $\sin\left(\frac{1}{x_n}\right) \to 0, \sin\left(\frac{1}{y_n}\right) \to 1$. In fact, for every $c \in [-1, 1]$, we can find a sequence z_n such that $z_n \to 0$ and $\sin(\frac{1}{z_n}) \to c$ as $n \to \infty$.

By now we are familiar with limits and one can expect the following:

Theorem 1.0.4. *Suppose* $\lim_{x\to c} f(x) = L$ *and* $\lim_{x\to c} g(x) = M$*, then*

1. $\lim_{x\to c}(f(x) \pm g(x)) = L \pm M$.
2. $\lim_{x\to c}(fg)(x) = LM$ *and when* $M \neq 0$, $\lim_{x\to c} \frac{f}{g}(x) = \frac{L}{M}$.
3. *(Sandwich): Suppose that* $h(x)$ *satisfies* $f(x) \leq h(x) \leq g(x)$ *in an interval containing* c*, and* $L = M$*. Then* $\lim_{x\to c} h(x) = L$.

Proof. We give the proof of (ii). Proof of other assertions are easy to prove. Let $\epsilon > 0$. From the definition of limit, we have $\delta_1, \delta_2, \delta_3 > 0$ such that

$$|x - c| < \delta_1 \implies |f(x) - L| < \frac{1}{2} \implies |f(x)| < N \text{ for some } N > 0,$$

$$|x - c| < \delta_2 \implies |f(x) - L| < \frac{\epsilon}{2M}, \text{ and}$$

$$|x - c| < \delta_3 \implies |g(x) - M| < \frac{\epsilon}{2N}.$$

Hence for $|x - c| < \delta = \min\{\delta_1, \delta_2, \delta_3\}$, we have

$$\begin{aligned} |f(x)g(x) - LM| &\leq |f(x)g(x) - f(x)M| + |f(x)M - LM| \\ &\leq |f(x)||g(x) - M| + M|f(x) - L| < \epsilon. \end{aligned}$$

To prove the second part, W.L.G assume $M > 0$. Then note that there exists an interval $(c - \delta_1, c + \delta_1)$ around c such that $g(x) > \frac{M}{2}$ in $(c - \delta_1, c + \delta_1)$. Then the back calculation

$$\left|\frac{1}{g(x)} - \frac{1}{M}\right| \leq \frac{|g(x) - M|}{M|g(x)|} \leq \frac{2|g(x) - M|}{M^2}$$

in $(c - \delta_1, c + \delta_1)$. Now from the definition of the limit there exists δ_2 such that

$$|x - c| < \delta_2 \implies |g(x) - M| < \frac{M^2\epsilon}{2}$$

Therefore taking $\delta = \min\{\delta_1, \delta_2\}$, from the above two steps we get

$$|x - c| < \delta \implies \left|\frac{1}{g(x)} - \frac{1}{M}\right| \leq \epsilon.$$

Examples 1.0.5. (i) $\lim_{x\to 0} x^m = 0$ $(m > 0)$. (ii) $\lim_{x\to 0} x \sin x = 0$.

Remark 1.2. *Suppose* $f(x)$ *is bounded in an interval containing* c *and* $\lim_{x\to c} g(x) = 0$*. Then* $\lim_{x\to c} f(x)g(x) = 0$.

Examples 1.0.6. (i) $\lim_{x\to 0} |x| \sin \frac{1}{x} = 0$. (ii) $\lim_{x\to 0} |x| \ln(1 + |x|) = 0$.