Lecture 13

1 Limits ctd..

One sided limits: Let $f(x)$ is defined on (c, b). The right hand limit of $f(x)$ at c is L, if given $\epsilon > 0$, there exists $\delta > 0$, such that

$$0 < x - c < \delta \implies |f(x) - L| < \epsilon.$$

Notation: $\lim_{x \to c^+} f(x) = L$. Similarly, one can define the left hand limit of $f(x)$ at b and is denoted by $\lim_{x \to b^-} f(x) = L$.

Both theorems above holds for right and left limits. Proof is easy.

Problem: Show that $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$.
Solution: Consider the unit circle centered at $O(0, 0)$ and passing through $A(1, 0)$ and $B(0, 1)$. Let Q be the projection of P on $x-$axis and the point T is such that A is the projection of T on $x-$axis. Let OT be the ray with $\angle AOT = \theta, 0 < \theta < \pi/2$. Let P be the point of intersection of OT and circle. Then ΔOPQ and ΔOTA are similar triangles and hence, Area of $\Delta OAP <$ Area of sector $OAP <$ area of ΔOAT. i.e.,

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$$

dividing by $\sin\theta$, we get $1 > \dfrac{\sin\theta}{\theta} > \cos\theta$. Now $\lim_{\theta \to 0^+} \cos\theta = 1$ implies that $\lim_{\theta \to 0^+} \dfrac{\sin\theta}{\theta} = 1$.
Now use the fact that $\dfrac{\sin\theta}{\theta}$ is even function.
At this stage, it is not difficult to prove the following:

Theorem 1.0.1. *$\lim_{x \to a} f(x) = L$ exists $\iff$ $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$.*

Limits at infinity and infinite limits

Definition 1.0.2. *$f(x)$ has limit L as x approaches $+\infty$, if for any given $\epsilon > 0$, there exists $M > 0$ such that*

$$x > M \implies |f(x) - L| < \epsilon.$$

Similarly, one can define limit as x approaches $-\infty$.

Problem: (i) $\lim_{x\to\infty}\dfrac{1}{x}=0$, (ii) $\lim_{x\to-\infty}\frac{1}{x}=0$. (iii) $\lim_{x\to\infty}\sin x$ does not exist.
Solution: (i) and (ii) are easy. For (iii), Choose $x_n=n\pi$ and $y_n=\frac{\pi}{2}+2n\pi$. Then $x_n,y_n\to\infty$ and $\sin x_n=0$, $\sin y_n=1$. Hence the limit does not exist.

Above two theorems on limits hold in this also.

Definition 1.0.3. *(Horizontal Asymptote:) A line $y=b$ is a horizontal asymptote of $y=f(x)$ if either $\lim_{x\to\infty}f(x)=b$ or $\lim_{x\to-\infty}f(x)=b$.*

Example 1.0.4. (i) *$y=1$ is a horizontal asymptote for $1+\frac{1}{x+1}$*

Definition 1.0.5. *(Infinite Limit): A function $f(x)$ approaches ∞ ($f(x)\to\infty$) as $x\to x_0$ if, for every real $B>0$, there exists $\delta>0$ such that*

$$0<|x-x_0|<\delta \implies f(x)>B.$$

Similarly, one can define for $-\infty$. Also one can define one sided limit of $f(x)$ approaching ∞ or $-\infty$.

Examples 1.0.6. (i) $\lim_{x\to0}\dfrac{1}{x^2}=\infty$, (ii) $\lim_{x\to0}\frac{1}{x^2}\sin(\frac{1}{x})$ *does not exist.*

For (i) given $B>0$, we can choose $\delta\le\frac{1}{\sqrt{B}}$. For (ii), choose a sequence $\{x_n\}$ such that $\sin\frac{1}{x_n}=1$, say $\frac{1}{x_n}=\frac{\pi}{2}+2n\pi$ and $\frac{1}{y_n}=n\pi$. Then $\lim_{n\to\infty}f(x_n)=\dfrac{1}{x_n^2}\to\infty$ and $\lim_{n\to\infty}f(y_n)=0$, though $x_n,y_n\to0$ as $n\to\infty$.

Definition 1.0.7. *(Vertical Asymptote:) A line $x=a$ is a vertical asymptote of $y=f(x)$ if either $\lim_{x\to a^+}f(x)=\pm\infty$ or $\lim_{x\to a^-}f(x)=\pm\infty$.*

Example 1.0.8. $f(x)=\frac{x+3}{x+2}$.
$x=-2$ is a vertical asymptote and $y=1$ is a horizontal asymptote.

1.1 Continuous functions

Definition 1.1.1. *A real valued function $f(x)$ is said to be continuous at $x=c$ if*
(i) $c\in domain(f)$
(ii) $\lim_{x\to c}f(x)$ exists
(iii) The limit in (ii) is equal to $f(c)$.

In other words, for every sequence $x_n \to c$, we must have $f(x_n) \to f(c)$ as $n \to \infty$. That is, for a given $\epsilon > 0$, there exists $\delta > 0$ such that

$$|x - c| < \delta \implies |f(x) - f(c)| < \epsilon.$$

Examples 1.1.2. *1.* $f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$ *is continuous at* 0.

Let $\epsilon > 0$. *Then* $|f(x) - f(0)| \leq |x^2|$. *So it is enough to choose* $\delta = \sqrt{\epsilon}$.

2. $g(x) = \begin{cases} \frac{1}{x} \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ *is not continuous at* 0.

Choose $\frac{1}{x_n} = \frac{\pi}{2} + 2n\pi$. *Then* $\lim x_n = 0$ *and* $f(x_n) = \frac{1}{x_n} \to \infty$.

The following theorem is an easy consequence of the definition.

Theorem 1.1.3. *Suppose* f *and* g *are continuous at* c. *Then*
(i) $f \pm g$ *is also continuous at* c
(ii) fg *is continuous at* c
(iii) $\dfrac{f}{g}$ *is continuous at* c *if* $g(c) \neq 0$.

Theorem 1.1.4. *Composition of continuous functions is also continuous i.e., if* f *is continuous at* c *and* g *is continuous at* $f(c)$ *then* $g(f(x))$ *is continuous at* c.

Corollary 1.1.5. *If* $f(x)$ *is continuous at* c, *then* $|f|$ *is also continuous at* c.

Theorem 1.1.6. *If* f, g *are continuous at* c, *then* $\max(f, g)$ *is continuous at* c.

Proof. Proof follows from the relation

$$\max(f, g) = \frac{1}{2}(f + g) + \frac{1}{2}|f - g|$$

and the theorems on algebra of limits. ///

Remark 1.1. *For sequences: If* f *is continuous function and let* $x_n \to x_0$ *then* $f(x_n) \to f(x_0)$ *(here we assumed that* $x_n, x_0 \in dom(f)$*)*

Example 1.1.7. *1.* $(1+1/n)^n \to e$ *implies* $(1+1/n)^{2n} \to e^2$ *as* $f(x) = x^2$ *is continuous on* $\mathbb{R}$.

2. $n^{1/n} \to 1$ *implies* $\frac{n^{1/n}}{1+n^{1/n}} \to 1/2$ *as* $f(x) = \frac{x}{1+x}$ *is continuous for all* $x > 0$.