Lecture 14

Types of discontinuities

Removable discontinuity: $f(x)$ is defined every where in an interval containing a except at $x = a$ and limit exists at $x = a$ OR $f(x)$ is defined also at $x = a$ and limit is NOT equal to function value at $x = a$. Then we say that $f(x)$ has removable discontinuity at $x = a$. These functions can be extended as continuous functions by defining the value of f to be the limit value at $x = a$.

Example 0.0.1. $f(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$. *Here limit as $x \to 0$ is 1. But $f(0)$ is defined to be 0.*

Jump discontinuity: The left and right limits of $f(x)$ exists but not equal. This type of discontinuities are also called discontinuities of first kind.

Example 0.0.2. $f(x) = \begin{cases} 1 & x \leq 0 \\ -1 & x \geq 0 \end{cases}$. *Easy to see that left and right limits at 0 are different.*

Infinite discontinuity: Left or right limit of $f(x)$ is ∞ or $-\infty$.

Example 0.0.3. $f(x) = \frac{1}{x}$ *has infinite discontinuity at $x = 0$.*

Discontinuity of second kind: If either $\lim\limits_{x\to c^-} f(x)$ or $\lim\limits_{x\to c^+} f(x)$ does not exist, then c is called discontinuity of second kind.

Example 0.0.4. *Consider the function*

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q} \end{cases}$$

Then f does not have left or right limit any point c. Indeed, if $c \in \mathbb{Q}$, then $x_n = c + \frac{1}{n} \in \mathbb{Q}$ and $y_n = c + \frac{\pi}{n} \notin \mathbb{Q}$. For these sequences, we will have, $f(c + \frac{1}{n})$ and $f(c + \frac{\pi}{n})$ converges to different values. If $c \notin \mathbb{Q}$, then choos $x_n \in (c, c + \frac{1}{n}) \cap \mathbb{Q}$ and $y_n = c + \frac{1}{n} \notin \mathbb{Q}$. For these sequences we will again get different limit values.

Properties of continuous functions

Definition 0.0.5. *(Closed set): A subset A of $\mathbb{R}$ is called closed set if A contains all its limit points. (i.e., if $\{x_n\} \subset A$ and $x_n \to c$, then $c \in A$).*

Theorem 0.0.6. *Continuous functions on closed, bounded interval is bounded.*

Proof. Let $f(x)$ be continuous on $[a, b]$ and let $\{x_n\} \subset [a, b]$ be a sequence such that $|f(x_n)| > n$. Then $\{x_n\}$ is a bounded sequence and hence there exists a subsequence $\{x_{n_k}\}$ which converges to c. Then $f(x_{n_k}) \to f(c)$, a contradiction to $|f(x_{n_k}| > n_k$.

Theorem 0.0.7. *Let $f(x)$ be a continuous function on closed, bounded interval $[a, b]$. Then supremum and infimum of functions are achieved in $[a, b]$.*

Proof. Above theoem and the completeness of $\mathbb{R}$ implies that Supremum of f is finite Let $\{x_n\}$ be a sequence such that $f(x_n) \to \sup f$. Then $\{x_n\}$ is bounded and hence by Bolzano-Weierstrass theorem, there exists a subsequence x_{n_k} such that $x_{n_k} \to x_0$ for some x_0. $a \leq x_n \leq b$ implies $x_0 \in [a, b]$. Since f is continuous, $f(x_{n_k}) \to f(x_0)$. Hence $f(x_0) = \sup f$. The attainment of minimum can be proved by noting that $-f$ is also continuous and $\min f = -\sup(-f)$.

Remark 0.1. *Closed and boundedness of the interval is important in the above theorem. Consider the examples (i) $f(x) = \frac{1}{x}$ on $(0, 1)$ (ii) $f(x) = x$ on $\mathbb{R}$.*

Theorem 0.0.8. *Let $f(x)$ be a continuous function on $[a, b]$ and let $f(c) > 0$ for some $c \in (a, b)$, Then there exists $\delta > 0$ such that $f(x) > 0$ in $(c - \delta, c + \delta)$.*

Proof. Let $\epsilon = \frac{1}{2}f(c) > 0$. Since $f(x)$ is continuous at c, there exists $\delta > 0$ such that

$$|x - c| < \delta \implies |f(x) - f(c)| < \frac{1}{2}f(c)$$

i.e., $-\frac{1}{2}f(c) < f(x) - f(c) < \frac{1}{2}f(c)$. Hence $f(x) > \frac{1}{2}f(c)$ for all $x \in (c - \delta, c + \delta)$.

Corollary 0.0.9. *Suppose a continuous functions $f(x)$ satisfies $\int_a^b f(x)\phi(x)dx = 0$ for all continuous functions $\phi(x)$ on $[a, b]$. Then $f(x) \equiv 0$ on $[a, b]$.*

Proof. Suppose $f(c) > 0$. Then by above theorem $f(x) > 0$ in $(c - \delta, c + \delta)$. Choose $\phi(x)$ so that $\phi(x) > 0$ in $(c - \delta/2, c + \delta/2)$ and is 0 otherwise. Then $\int_a^b f(x)\phi(x) > 0$. A contradiction.
Alternatively, one can choose $\phi(x) = f(x)$.

Theorem 0.0.10. *Let $f(x)$ be a continuous function on $\mathbb{R}$ and let $f(a)f(b) < 0$ for some a, b. Then there exits $c \in (a, b)$ such that $f(c) = 0$.*

Proof. Assume that $f(a) < 0 < f(b)$. Let $S = \{x \in [a, b] : f(x) < 0\}$. Then $[a, a+\delta) \subset S$ for some $\delta > 0$ and S is bounded. Let $c = \sup S$. We claim that $f(c) = 0$. Take $x_n = c+\frac{1}{n}$, then $x_n \notin S$, $x_n \to c$. Therefore, $f(c) = \lim f(x_n) \geq 0$. On the otherhand, note that $c - \frac{1}{n}$ is NOT supremum. Therefore, there exists a point $y_n \in (c - \frac{1}{n}, c) \cap S$. Then note that $y_n \to c$, $f(c) = \lim f(y_n) \leq 0$. Hence $f(c) = 0$.

Corollary 0.0.11. Intermediate value theorem: *Let $f(x)$ be a continuous function on $[a, b]$ and let $f(a) < y < f(b)$. Then there exists $c \in (a, b)$ such that $f(c) = y$*

Remark 0.2. *A continuous function assumes all values between its maximum and minimum.*

Problem: (fixed point): Let $f(x)$ be a continuous function from $[0, 1]$ into $[0, 1]$. Then show that there is a point $c \in [0, 1]$ such that $f(c) = c$.

Define the function $g(x) = f(x) - x$. Then $g(0) \geq 0$ and $g(1) \leq 0$. Now Apply Intermediate value theorem.

Application: Root finding: To find the solutions of $f(x) = 0$, one can think of defining a new function g such that $g(x)$ has a fixed point, which in turn satisfies $f(x) = 0$.
Example: (1) $f(x) = x^3 + 4x^2 - 10$ in the interval $[1, 2]$. Define $g(x) = \left(\frac{10}{4+x}\right)^{1/2}$. We can check that g maps $[1, 2]$ into $[1, 2]$. So g has fixed point in $[1, 2]$ which is also solution of $f(x) = 0$. Such fixed points can be obtained as limit of the sequence $\{x_n\}$, where $x_{n+1} = g(x_n), x_0 \in (1, 2)$. Note that

$$g'(x) = \frac{\sqrt{10}}{(4 + x)^{3/2}} < \frac{1}{2}.$$

By Mean Value Theorem, $\exists\ z$ (see next section)

$$|x_{n+1} - x_n| = |g'(z)||x_n - x_{n-1}| \leq \frac{1}{2}|x_n - x_{n-1}|$$

Iterating this, we get

$$|x_{n+1} - x_n| < \frac{1}{2^n}|x_1 - x_0|.$$

Therefore, $\{x_n\}$ is a Cauchy sequence. (see problem after Theorem 1.4.4).