Lecture 15

1 Uniformly continuous functions

Definition 1.0.1. *A function $f(x)$ is said to be uniformly continuous on a set S, if for given $\epsilon > 0$, there exists $\delta > 0$ such that*

$$x, y \in S, \quad |x - y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

Here δ depends only on ϵ, not on x or y.

Examples 1.0.2. *1. $f(x) = x$ is uniformly continuous on $\mathbb{R}$. In this case we can choose $\delta < \epsilon$.*

2. $f(x) = x^2$ is uniformly continuous on bounded interval $[a, b]$.
Note that $|x^2 - y^2| \leq |x + y||x - y| \leq 2b|x - y|$. So one can choose $\delta < \frac{\epsilon}{2b}$.

Remark 1.1. *In case of continuous function $f(x) = \frac{1}{x}$ on $(0, 1)$, geometrically one can visualize that the value of δ for the same ϵ depends on the point. Moreover the δ becomes smaller and smaller as we take the point close to 0. But this is only intuition. For getting a mathematical proof, we need the following*

Proposition 1.0.3. *Proposition: If $f(x)$ is uniformly continuous function $\iff$ for ANY two sequences $\{x_n\}, \{y_n\}$ such that $|x_n - y_n| \to 0$, we have $|f(x_n) - f(y_n)| \to 0$ as $n \to \infty$.*

Proof. Suppose not. Then there exists $\{x_n\}, \{y_n\}$ such that $|x_n - y_n| \to 0$ and $|f(x_n) - f(y_n)| > \eta$ for some $\eta > 0$. Then it is clear that for $\epsilon = \eta$, there is no δ for which $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$. Because the above sequence satisfies $|x - y| < \delta$, but its image does not.
For converse, assume that for any two sequences $\{x_n\}, \{y_n\}$ such that $|x_n - y_n| \to 0$ we have $|f(x_n) - f(y_n)| \to 0$. Suppose f is not uniformly continuous. Then by the definition there exists ϵ_0 such that for any $\delta > 0$, there exist $z, w \in |x - y| < \delta$ for which $|f(z) - f(w)| > \epsilon_0$. Now take $\delta = \frac{1}{n}$ and choose z_n, w_n such that $|z_n - w_n| < \frac{1}{n}$. Then $|f(z_n) - f(w_n)| > \epsilon_0$. A contradiction.

Examples 1.0.4. *1. $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0, 1)$.*
Take $x_n = \frac{1}{n+1}, y_n = \frac{1}{n}$, then for n large $|x_n - y_n| \to 0$ but $|f(x_n) - f(y_n)| = 1$.

2. $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.
Take $x_n = n + \frac{1}{n}$ and $y_n = n$. Then $|x_n - y_n| = \frac{1}{n} \to 0$, but $|f(x_n) - f(y_n)| = 2 + \frac{1}{n^2} > 2$.

Remark 1.2. *1. It is easy to see from the definition that if f, g are uniformly continuous, then $f \pm g$ is also uniformly continuous.*

2. If f, g are uniformly continuous, then fg need not be uniformly continuous. This can be seen by noting that $f(x) = x$ is uniformly continuous on $\mathbb{R}$ but x^2 is not uniformly continuous on $\mathbb{R}$.

Theorem 1.0.5. *A continuous function $f(x)$ on a closed, bounded interval $[a, b]$ is uniformly continuous.*

Proof. Suppose not. Then there exists $\epsilon > 0$ and sequences $\{x_n\}$ and $\{y_n\}$ in $[a, b]$ such that $|x_n - y_n| < \frac{1}{n}$ and $|f(x_n) - f(y_n)| > \epsilon$. But then by Bolzano-Weierstrass theorem, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges to x_0. Also $y_{n_k} \to x_0$. Now since f is continuous, we have $f(x_0) = \lim f(x_{n_k}) = \lim f(y_{n_k})$. Hence $|f(x_{n_k}) - f(y_{n_k})| \to 0$, a contradiction.

Example 1.0.6. $f(x) = \sqrt{x}$ *is uniformly continuous on* $[0, 1]$

Using mean value theorem of differentiable functions we will show (in lecture 17) that $f(x) = \sqrt{x}, x \in [1, \infty)$ is uniformly continuous. Combining these two intervals we can easily see that $\sqrt{x}$ is uniformly continuous on $[0, \infty)$.

Example 1.0.7. $f(x) = x \sin x$ *and* $f(x) = \sin x^2$ *are uniformly continuous on any closed bounded interval* $[a, b]$

Corollary 1.0.8. *Suppose $f(x)$ has only removable discontinuities in $[a, b]$. Then $\tilde{f}$, the extension of f, is uniformly continuous.*

Example 1.0.9. $f(x) = \frac{\sin x}{x}$ *for* $x \neq 0$ *and* 0 *for* $x = 0$ *on* $[0, 1]$ *has extension* $\tilde{f}(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$ *which is uniformly continuous on* $[-1, 1]$.

Theorem 1.0.10. *Let f be a uniformly continuous function and let $\{x_n\}$ be a cauchy sequence. Then $\{f(x_n)\}$ is also a Cauchy sequence.*

Proof. Let $\epsilon > 0$. As f is uniformly continuous, there exists $\delta > 0$ such that

$$|x - y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

Since $\{x_n\}$ is a Cauchy sequence, there exists N such that

$$m, n > N \implies |x_n - x_m| < \delta.$$

Therefore $|f(x_n) - f(x_m)| < \epsilon$ for all $m, n > N$.

Corollary 1.0.11. *Let I be an open interval $f : I \to \mathbb{R}$ be a uniformly continuous function. Let a be an end point of I. Then limit $\lim_{x \to a} f(x)$ exists.*

Example 1.0.12. $f(x) = \frac{1}{x^2}$ *is not uniformly continuous on* $(0, 1)$.

The sequence $x_n = \frac{1}{n}$ is cauchy but $f(x_n) = n^2$ is not. Hence f cannot be uniformly continuous.

Example 1.0.13. $f(x) = \sin(1/x), x \in (0, 1)$ *is not uniformly continuous.*