Lecture 16

1 Differentiability

Definition 1.0.1. *A real valued function $f(x)$ is said to be differentiable at x_0 if*

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} \quad \textit{exists.}$$

This limit is called the derivative of f at x_0, denoted by $f'(x_0)$.

Example: $f(x)=x^2$

$$f'(x)=\lim_{h\to 0}\frac{2xh+h^2}{h}=2x.$$

Theorem 1.0.2. *If $f(x)$ is differentiable at a, then it is continuous at a.*

Proof. For $x\neq a$, we may write,

$$f(x)=(x-a)\frac{f(x)-f(a)}{(x-a)}+f(a).$$

Now taking the limit $x\to a$ and noting that $\lim(x-a)=0$ and $\lim\frac{f(x)-f(a)}{(x-a)}=f'(a)$, we get the result.

Theorem 1.0.3. *Let f,g be differentiable at $c\in(a,b)$. Then $f\pm g, fg$ and $\frac{f}{g}$ $(g(c)\neq 0)$ is also differentiable at c*

Proof. We give the proof for product formula: First note that

$$\frac{(fg)(x)-(fg)(c)}{x-c}=f(x)\frac{g(x)-g(c)}{x-c}+g(c)\frac{f(x)-f(c)}{x-c}.$$

Now taking the limit $x\to c$, we get the product formula

$$(fg)'(c)=f(c)g'(c)+f'(c)g(c).$$

Since $g(c)\neq 0$ and g is continuous, we get $g(x)\neq 0$ in a small interval around c. Therefore

$$\frac{f}{g}(x)-\frac{f}{g}(c)=\frac{g(c)f(x)-g(c)f(c)+g(c)f(c)-g(x)f(c)}{g(x)g(c)}$$

Hence

$$\frac{(f/g)(x)-(f/g)(c)}{x-c}=\left\{g(c)\frac{f(x)-f(c)}{x-c}-f(c)\frac{g(x)-g(c)}{x-c}\right\}\frac{1}{g(x)g(c)}$$

Now taking the limit $x \to c$, we get

$$(\frac{f}{g})'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{g^2(c)}.$$

Theorem 1.0.4. *(Chain Rule): Suppose $f(x)$ is differentiable at c and g is differentiable at $f(c)$, then $h(x) := g(f(x))$ is differentiable at c and*

$$h'(c) = g'(f(c))f'(c)$$

Proof. Define the function h as

$$h(y) = \begin{cases} \frac{g(y)-g(f(c))}{y-f(c)} & y \neq f(c) \\ g'(f(c)) & y = f(c) \end{cases}$$

Then the function h is continuous at $y = f(c)$ and $g(y) - g(f(c)) = h(y)(y - f(c))$, so

$$\frac{g(f(x)) - g(f(c))}{x - c} = h(f(x))\frac{f(x) - f(c)}{x - c}.$$

Now taking limit $x \to c$, we get the required formula.

Local extremum: A point $x = c$ is called local maximum of $f(x)$, if there exists $\delta > 0$ such that

$$0 < |x - c| < \delta \implies f(c) \geq f(x).$$

Similarly, one can define local minimum: $x = b$ is a local minimum of $f(x)$ if there exists $\delta > 0$ such that

$$0 < |x - b| < \delta \implies f(b) \leq f(x).$$

Theorem 1.0.5. *Let $f(x)$ be a differentiable function on (a, b) and let $c \in (a, b)$ is a local maximum of f. Then $f'(c) = 0$.*

Proof. Let δ be as in the above definition. Then

$$x \in (c, c + \delta) \implies \frac{f(x) - f(c)}{x - c} \leq 0$$

$$x \in (c - \delta, c) \implies \frac{f(x) - f(c)}{x - c} \geq 0.$$

Now taking the limit $x \to c$, we get $f'(c) = 0$.

Theorem 1.0.6. *Rolle's Theorem: Let $f(x)$ be a continuous function on $[a, b]$ and differentiable on (a, b) such that $f(a) = f(b)$. Then there exists $c \in (a, b)$ such that $f'(c) = 0$.*

Proof. If $f(x)$ is constant, then it is trivial. Suppose $f(x_0) > f(a)$ for some $x_0 \in (a, b)$, then f attains maximum at some $c \in (a, b)$. Other possibilities can be worked out similarly.

Theorem 1.0.7. *Mean-Value Theorem (MVT): Let f be a continuous function on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that*

$$f(b) - f(a) = f'(c)(b - a).$$

Proof. Let $l(x)$ be a straight line joining $(a, f(a))$ and $(b, f(b))$. Consider the function $g(x) = f(x) - l(x)$. Then $g(a) = g(b) = 0$. Hence by Rolle's theorem

$$0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}$$

Corollary 1.0.8. *If f is a differentiable function on (a, b) and $f' = 0$, then f is constant.*

Proof. By mean value theorem $f(x) - f(y) = 0$ for all $x, y \in (a, b)$.

Remark 1.1. *If $f(x)$ is differentiable and $\sup |f'(x)| < C$ for some C. Then, f is uniformly continuous.*
Apply mean value theorem to get $|f(x) - f(y)| \le C|x - y|$ for all x, y. Hence given ϵ, we may choose δ to be less than ϵ/C.

Example 1.0.9. *Show that $\cos x$ is uniformly continuous on $\mathbb{R}$*
By MVT, we get

$$|\cos x - \cos y| \le |\sin c||x - y| \le |x - y|$$

therefore for any given ϵ we may choose any $\delta < \epsilon$. ///

Remark 1.2. *We note that if f, g satisfies*

$$|f(x) - f(y)| \le C_1|x - y|; \; |g(x) - g(y)| \le C_2|x - y|, \forall x, y$$

Then $h(x) = f(g(x))$ also satisfies

$$|h(x) - h(y)| \le C_1C_2|x - y|, \; \forall x, y$$

As a consequence, we can easily see that

Example 1.0.10. *$f(x) = |\sin x|, \; x \in \mathbb{R}$ is uniformly continuous.*

Example 1.0.11. *$f(x) = \sqrt{\log x}, \; x > 2$ is uniformly continuous.*

Remark 1.3. *suppose we have a sequence defined by* $x_{n+1} = f(x_n), n \geq 1, x_0 \in$ *domain of* f *and if* $|f'| \leq \alpha < 1$, *then* $\{x_n\}$ *converges.*
As a consequence of the MVT,

$$|x_{n+1} - x_n| \leq |f(x_n) - f(x_{n-1})| \leq |f'(c)||x_n - x_{n-1}| < \alpha|x_n - x_{n-1}|$$

Therefore $\{x_n\}$ *is a Cauchy sequence and hence converges.*

Example 1.0.12. *Let* $g(x) = \left(\frac{10}{4+x}\right)^{1/2}$ *on* $[1, 2]$ *and the sequence defined by* $x_1 \in (1, 2)$, $x_{n+1} = g(x_n), n \in \mathbb{N}$ *converges.*

Definition 1.0.13. *A function* $f(x)$ *is strictly increasing on an interval* I, *if for* $x, y \in I$ *with* $x < y$ *we have* $f(x) < f(y)$. *We say* f *is strictly decreasing if* $x < y$ *in* I *implies* $f(x) > f(y)$.

Theorem 1.0.14. *A differentiable function* f *is (i) strictly increasing in* (a, b) *if* $f'(x) > 0$ *for all* $x \in (a, b)$. *(ii) strictly decreasing in* (a, b) *if* $f'(x) < 0$.

Proof. Choose x, y in (a, b) such $x < y$. Then by MVT, for some $c \in (x, y)$

$$\frac{f(x) - f(y)}{x - y} = f'(c) > 0.$$

Hence $f(x) < f(y)$.

Example 1.0.15. *Find the intervals where the following function is increasing and decreasing*

$$f(x) = x^2(x - \frac{3}{2})$$