Lecture 17

1 Taylor's theorem

Let f be a k times differentiable function on an interval I of $\mathbb{R}$. We want to approximate this function by a polynomial $P_n(x)$ such that $P_n(a) = f(a)$ at a point a. Moreover, if the derivatives of f and P_n also equal at a then we see that this approximation becomes more accurate in a neighbourhood of a. So the best coefficients of the polynomial can be calculated using the relation $f^{(k)}(a) = P_n^{(k)}(a), k = 0, 1, 2, ..., n$. The best is in the sense that if $f(x)$ itself is a polynomial of degree less than or equal to n, then both f and P_n are equal. This implies that the polynomial is $\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$. Then we write $f(x) = P_n(x) + R_n(x)$ in a neighbourhood of a. From this, we also expect the $R_n(x) \to 0$ as $x \to a$. In fact, we have the following theorem known as **Taylor's theorem**:

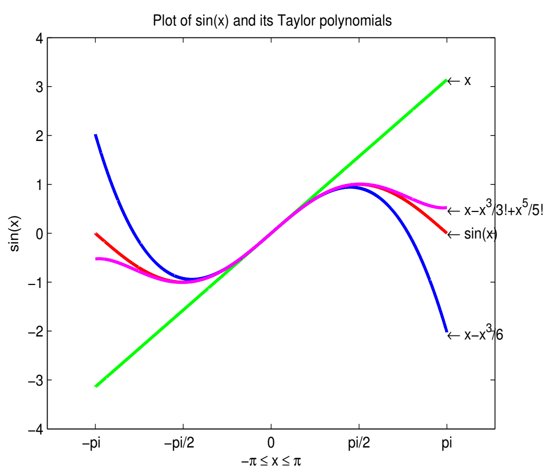

Figure 1: Approximation of sin(x) by Taylor's polynomials

Theorem 1.0.1. *Let $f(x)$ and its derivatives of order m are continuous and $f^{(m+1)}(x)$ exists in a neighbourhood of $x = a$. Then there exists $c \in (a, x)($ or $c \in (x, a))$ such that*

$$f(x) = f(a) + f'(a)(x-a) + ... + f^{(m)}(a)\frac{(x-a)^m}{m!} + R_m(x)$$

where $R_m(x) = \frac{f^{(m+1)}(c)}{(m+1)!}(x-a)^{m+1}$.

Proof. Define the functions F and g as

$$F(y) = f(x) - f(y) - f'(y)(x-y) - .... - \frac{f^{(m)}(y)}{m!}(x-y)^m,$$

$$g(y) = F(y) - \left(\frac{x-y}{x-a}\right)^{m+1} F(a).$$

Then it is easy to check that $g(a) = 0$. Also $g(x) = F(x) = f(x) - f(x) = 0$. Therefore, by Rolle's theorem, there exists some $c \in (a, x)$ such that

$$g'(c) = 0 = F'(c) + \frac{(m+1)(x-c)^m}{(x-a)^{m+1}} F(a).$$

On the other hand, from the definition of F,

$$F'(c) = -\frac{f^{(m+1)}(c)}{m!}(x-c)^m.$$

Hence $F(a) = \dfrac{(x-a)^{m+1}}{(m+1)!} f^{(m+1)}(c)$ and the result follows.

Examples 1.0.2. *(i)* $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!}e^c, c \in (0, x)$ *or* $(x, 0)$ *depending on the sign of* x.
(ii) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + ... + \frac{x^n}{n!}\sin(c + \frac{n\pi}{2}), c \in (0, x)$ *or* $(x, 0)$.
(iii) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + ... + \frac{x^n}{n!}\cos(c + \frac{n\pi}{2}), c \in (0, x)$ *or* $(x, 0)$.

Example 1.0.3. *Find the order* n *of Taylor Polynomial* P_n, *about* $x = 0$ *to approximate* e^x *in* $(-1, 1)$ *so that the error is not more than* 0.005

Solution: We know that $p_n(x) = 1 + x + ... + \frac{x^n}{n!}$. The maximum error in [-1,1] is

$$|R_n(x)| \leq \frac{1}{(n+1)!} \max_{[-1,1]} |x|^{n+1} e^x \leq \frac{e}{(n+1)!}.$$

So n is such that $\frac{e}{(n+1)!} \leq 0.005$ or $n \geq 5$.

Example 1.0.4. *Find the interval of validity when we approximate* $\cos x$ *with 2nd order polynomial with error tolerance* 10^{-4}.

Solution: Taylor polynomial of degree 2 for $\cos x$ is $1 - \frac{x^2}{2}$. So the remainder is $(\sin c)\frac{x^3}{3!}$. Since $|\sin c| \leq 1$, the error will be atmost 10^{-4} if $|\frac{x^3}{3!}| \leq 10^{-4}$. Solving this gives $|x| < 0.084$.