Lecture 18

1 Taylor's theorem ctd..

Maxima and Minima: Derivative test

Definition 1.0.1. *A point $x = a$ is called critical point of the function $f(x)$ if $f'(a) = 0$.*

Second derivative test: A point $x = a$ is a local maxima if $f'(a) = 0, f''(a) < 0$.

Suppose $f(x)$ is continuously differentiable in an interval around $x = a$ and let $x = a$ be a critical point of f. Then $f'(a) = 0$. By Taylor's theorem around $x = a$, there exists, $c \in (a, x)$ (or $c \in (x, a)$),

$$f(x) - f(a) = \frac{f''(c)}{2}(x - a)^2.$$

If $f''(a) < 0$. Then by the continuity of f'', there exists $\delta > 0$ such that $f''(c) < 0$ in $|x - a| < \delta$. Hence $f(x) < f(a)$ in $|x - a| < \delta$, which implies that $x = a$ is a local maximum.

Similarly, one can show the following for local minima: $x = a$ is a local minima if $f'(a) = 0, f''(a) > 0$.

Also the above observations show that if $f'(a) = 0, f''(a) = 0$ and $f^{(3)}(a) \neq 0$, then the sign of $f(x) - f(a)$ depends on $(x - a)^3$. i.e., it has no constant sign in any interval containing a. Such point is called point of inflection or saddle point.

We can also derive that if $f'(a) = f''(a) = f^{(3)}(a) = 0$, then we again have $x = a$ is a local minima if $f^{(4)}(a) > 0$ and is a local maxima if $f^{(4)}(a) < 0$.

Summarizing the above, we have:

Theorem 1.0.2. *Let f be a real valued function that is differentiable $2n$ times and $f^{(2n)}$ is continuous at $x = a$. Then*

1. *If $f^{(k)}(a) = 0$ for $k = 1, 2, ...2n - 1$ and $f^{(2n)}(a) > 0$ then a is a point of local minimum of $f(x)$*
2. *If $f^{(k)}(a) = 0$ for $k = 1, 2, ...2n - 1$ and $f^{(2n)}(a) < 0$ then a is a point local maximum of $f(x)$.*
3. *If $f^{(k)} = 0$ for $k = 1, 2, ...2n - 2$ and $f^{(2n-1)}(a) \neq 0$, then a is point of inflection. i.e., f has neither local maxima nor local minima at $x = a$.*

L'Hospitals Rule:
Suppose $f(x)$ and $g(x)$ are differentiable n times, $f^{(n)}, g^{(n)}$ are continuous at a and $f^{(k)}(a) =$

$g^{(k)}(a) = 0$ for $k = 0, 1, 2, ..., n-1$. Also if $g^{(n)}(a) \neq 0$. Then by Taylor's theorem,

$$\begin{aligned}\lim_{x \to a} \frac{f(x)}{g(x)} &= \lim_{x \to a} \frac{f^{(n)}(c)}{g^{(n)}(c)} \\ &= \frac{f^{(n)}(a)}{g^{(n)}(a)}\end{aligned}$$

In the above, we used the fact that $g^{(n)}(x) \neq 0$ "near $x = a$" and $g^{(n)}(c) \to g^{(n)}(a)$ as $x \to a$. Similarly, we can derive a formula for limits as x approaches infinity by taking $x = \frac{1}{y}$.

$$\begin{aligned}\lim_{x \to \infty} \frac{f(x)}{g(x)} &= \lim_{y \to 0} \frac{f(1/y)}{g(1/y)} \\ &= \lim_{y \to 0} \frac{(-1/y^2) f'(1/y)}{(-1/y^2) g'(1/y)} \\ &= \lim_{x \to \infty} \frac{f'(x)}{g'(x)}\end{aligned}$$

Taylor Series

Suppose f is infinitely differentiable at a and if the remainder term in the Taylor's formula, $R_n(x) \to 0$ as $n \to \infty$. Then we say that the Taylor Series converges to the function $f(x)$ at the point x. So we may write

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n. \tag{1.1}$$

For a fixed x, the above infinite sum is a series of real numbers. One can independently check the convergence of such series by the convergence tests. However that does not confirm if the series is equal to $f(x)$. The series on the RHS of (1.1) is called Taylor series of $f(x)$ about the point a.

In some cases it is easy to verify this. For example,

Suppose there exists $C = C(x) > 0$, independent of n, such that $|f^{(n)}(x)| \leq C(x)$. Then $|R_n(x)| \to 0$ if $\lim_{n \to \infty} \frac{|x-a|^{n+1}}{(n+1)!} = 0$. For any fixed x and a, we can always find N such that $|x - a| < N$. Let $q := \frac{|x-a|}{N} < 1$. Then

$$\begin{aligned}\left| \frac{(x-a)^{n+1}}{(n+1)!} \right| &= \left| \frac{|x-a|}{1} \right| \left| \frac{|x-a|}{2} \right| \cdots \left| \frac{|x-a|}{N-1} \right| \left| \frac{|x-a|}{N} \right| \cdots \left| \frac{|x-a|}{n+1} \right| \\ &< \left| \frac{|x-a|^{N-1}}{(N-1)!} \right| q^{n-N+2}\end{aligned}$$

$\to 0$ as $n \to \infty$ thanks to $q < 1$.

In case of $a = 0$, the formula obtained in Taylor's theorem is known as *Maclaurin's formula* and the corresponding series that one obtains is known as *Maclaurin's series.*

Examples 1.0.3. *1.* $f(x) = e^x$.

In this case $R_n(x) = \dfrac{x^{n+1}}{(n+1)!} f^{(n+1)}(c) = \dfrac{x^{n+1}}{(n+1)!} e^c = \dfrac{x^{n+1}}{(n+1)!} e^{\theta x}$, *for some* $\theta \in (0,1)$.

Therefore for any given x *fixed,* $\lim_{n\to\infty} |R_n(x)| = \lim_{n\to\infty} \left(\dfrac{x^{n+1}}{(n+1)!}\right) e^{\theta x} = 0.$

2. $f(x) = \sin x$.

In this case it is easy to see that $|R_n(x)| \leq \dfrac{|x|^{2n+1}}{(2n+1)!} |\sin(c + \dfrac{n\pi}{2})|$. *Now use the fact that* $|\sin x| \leq 1$ *and follow as in example* (i).

Definition 1.0.4. *An infinitely differentiable function* $f(x)$ *is called Real analytic at* $x = a$ *if the function has Taylor series expansion: There exists* $R > 0$ *and*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n, \quad |x-a| < R$$

Note that the series in the RHS may converge but not equal to $f(x)$. So to say that the function has its Taylor series we need to first check the remainder term $R_n(x)$ goes to zero as $n \to \infty$.

We state the following characterization of real analytic functions. The proof is beyond the scope of this course.

Theorem 1.0.5. *The following are equivalent*

1. $f(x)$ *is real analytic at* $x = a$
2. *For every small interval* I *containing* a, *there exist constants* $r > 0$ *and* $C > 0$ *such that for all* $k \in \mathbb{N} \cup \{0\}$:

$$\left| f^{(k)}(x) \right| \leq C \frac{k!}{r^k} \ \forall x \in I.$$

(2) $\implies$ (1): The remainder of Taylor's theorem R_n can be estimated as

$$\begin{aligned} |R_n(x)| &\leq \frac{C}{r^{n+1}} |x-a|^{n+1} \\ &= C \left(\frac{|x-a|}{r}\right)^{n+1} \to 0 \text{ if } |x-a| < r. \quad /// \end{aligned}$$

It is not easy to detect if a function is not real analytic. For example

$$f(x) = \begin{cases} e^{-1/x}, & x > 0 \\ 0 & x \leq 0. \end{cases}$$

This function is infinitely differentiable and all its derivatives at 0 are equal to zero. So the Taylor series at zero identically zero. But the function is not identically equal to zero in any interval containing zero.