Lecture 19

1 Power series

Motivation: One of the motivation to study power series from the integration point of view is when we want to find the value of integral

$$\int_0^1 x^x dx$$

From our understanding of functions we can see that x^x is continuous in $[0, 1]$. We may write

$$\int_0^1 x^x dx = \int_0^1 e^{x\log x} dx$$

Now using the Taylor series of exponential function, we may write

$$e^{x\log x} = \sum_{n=0}^{\infty} \frac{x^n(\log x)^n}{n!}$$

Therefore the integral becomes

$$\int_0^1 e^{x\log x} = \int_0^1 \sum_{n=0}^{\infty} \frac{x^n(\log x)^n}{n!} dx$$

So IF WE CAN interchange the operation of integration and infinite sum then this is equal to

$$\int_0^1 e^{x\log x} = \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^1 x^n(\log x)^n dx$$

This integral can be evaluated easily using integration by parts and one can get the value equal to

$$\int_0^1 x^x dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n^n}$$

This is an alternating series and converges (using Dirichlet test.) In this lecture our aim is to study the problem of the calculus of functions that are expressed as infinite series as above.

Definition 1.0.1. *Given a sequence of real numbers $\{a_n\}_{n=0}^{\infty}$, the series $\sum_{n=0}^{\infty} a_n(x-c)^n$ is called power series with center c.*

It is easy to see that a power series converges for $x = c$. Power series is a function of x

provided it converges for x. If a power series converges, then the domain of convergence is either a bounded interval or the whole of $\mathbb{R}$. So it is natural to study the largest interval where the power series converges.

Remark 1.1. *If $\sum a_n x^n$ converges at $x = r$, then $\sum a_n x^n$ converges for $|x| < |r|$.*

Proof. We can find $C > 0$ such that $|a_n x^n| \leq C$ for all n. Then

$$|a_n x^n| \leq |a_n r^n| |\frac{x}{r}|^n \leq C|\frac{x}{r}|^n.$$

Conclusion follows from comparison theorem.

Theorem 1.0.2. *Consider the power series $\sum_{n=0}^{\infty} a_n x^n$. Suppose $\beta = \limsup \sqrt[n]{|a_n|}$ and $R = \frac{1}{\beta}$ (We define $R = 0$ if $\beta = \infty$ and $R = \infty$ if $\beta = 0$). Then*

1. $\sum_{n=0}^{\infty} a_n x^n$ *converges for* $|x| < R$
2. $\sum_{n=0}^{\infty} a_n x^n$ *diverges for* $|x| > R$.
3. *No conclusion if* $|x| = R$.

Proof. Proof of (i) follows from the root test. For a proof, take $\alpha_n(x) = a_n x^n$ and $\alpha = \limsup \sqrt[n]{|\alpha_n|}$. For (ii), one can show that if $|x| > R$, then there exists a subsequence $\{a_n\}$ such that $a_n \not\to 0$. Notice that $\alpha = \beta|x|$. For (iii), observe as earlier that the series with $a_n = \frac{1}{n}$ and $b_n = \frac{1}{n^2}$ will have $R = 1$.
Similarly, we can prove:

Theorem 1.0.3. *Consider the power series $\sum_{n=0}^{\infty} a_n x^n$. Suppose $\beta = \lim \left|\frac{a_{n+1}}{a_n}\right|$ and $R = \frac{1}{\beta}$ (We define $R = 0$ if $\beta = \infty$ and $R = \infty$ if $\beta = 0$). Then*

1. $\sum_{n=0}^{\infty} a_n x^n$ *converges for* $|x| < R$
2. $\sum_{n=0}^{\infty} a_n x^n$ *diverges for* $|x| > R$.
3. *No conclusion if* $|x| = R$.

Definition 1.0.4. *The real number R in the above theorems is called the Radius of convergence of power series.*

Examples 1.0.5. *Find the interval of convergence of (i) $\sum \frac{x^n}{n}$ (ii) $\sum \frac{x^n}{n!}$ (iii) $\sum 2^{-n} x^{3n}$*

1. $\beta = \lim |\frac{a_{n+1}}{a_n}| = 1$, *and we know that the series does not converge for* $x = 1$, *but converges at* $x = -1$.

2. $\beta = \lim |\frac{a_{n+1}}{a_n}| = 0$. *Hence the series converges everywhere.*

3. *To see the subsequent non-zero terms, we write the series as* $\sum 2^{-n}(x^3)^n = \sum 2^{-n}y^n$. *For this series* $\beta_y = \limsup \sqrt[n]{|a_n|} = 2^{-1}$. *Therefore,* $\beta_x = 2^{-1/3}$ *and* $R = 2^{1/3}$.

Example 1.0.6. *Let* $a_n = \begin{cases} 2^n & n \text{ is even} \\ 2^{n-1} & n \text{ is odd} \end{cases}$. *Then* $\limsup \frac{a_{n+1}}{a_n} = 4$ *and the limit of* $\frac{a_{n+1}}{a_n}$ *does not exist. But* $\limsup |a_n|^{1/n} = 2$. *So the radius of convergence of the series* $\sum a_n x^n$ *is* $1/2$.

Theorem 1.0.7. Continuity of power series: *The function* f *defined as power series*

$$f(x) = \sum a_n x^n, |x| < R$$

is continuous at $x = 0$.

Proof. Proof follows from the following important estimate: For any $|x| < r < R$

$$|f(x) - a_0| \leq \sum_{n=1}^{\infty} |a_n||x|^n \leq |x| \sum_{n=1}^{\infty} |a_n||x|^{n-1} \leq |x| \sum_{n=1}^{\infty} |a_n||r|^{n-1}$$

Since the power series converges, it is not difficult to show for any $r < R$ the series $\sum_{n=1}^{\infty} |a_n||r|^{n-1}$ converges to S(say). Then

$$|f(x) - a_0| \leq |x|S \to 0. \text{ as } \; |x| \to 0.$$

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Theorem 1.0.8. *Suppose* $\sum a_n x^n = \sum b_n x^n$, $|x| < R$ *then* $a_n = b_n$ *for all* n.

Proof. By continuity, taking $x \to 0$, we get $a_0 = b_0$. Now assume by induction $a_n = b_n$ for all $n = 0, 1, 2, ...k$. Then by trivial cancellation on both sides we get

$$a_{k+1} + a_{k+2}x + ... = b_{k+1} + b_{k+2}x + ...$$

then again by continuity, we get $a_{k+1} = b_{k+1}$. ///

This also gives a very important method for solving differential equations. This we will see in the next lecture.