Lecture 2

1 Sequences and their limit

Definition 1.0.1. *A sequence of real numbers is a function from* $\mathbb{N}$ *to* $\mathbb{R}$.

Notation. *It is customary to denote a sequence as* $\{a_n\}_{n=1}^{\infty}$.

Examples 1.0.2. *(i)* $\{c\}_{n=1}^{\infty}, c \in \mathbb{R}$, *(ii)* $\{\frac{(-1)^{n+1}}{n}\}_{n=1}^{\infty}$, *(iii)* $\{\frac{n-1}{n}\}_{n=1}^{\infty}$ *and (iv)* $\{\sqrt{n}\}_{n=1}^{\infty}$.

Definition 1.0.3. *A sequence* $\{a_n\}_{n=1}^{\infty}$ *converges to limit* L *if for every* $\epsilon > 0$ *(given) there exists a positive integer* N *such that* $n \geq N \implies |a_n - L| < \epsilon$.

Notation. $L = \lim\limits_{n\to\infty} a_n$ *or* $a_n \to L$.

Examples 1.0.4.

(i) *It is clear that the constant sequence* $\{c\}_{n=1}^{\infty}, c \in \mathbb{R}$, *has* c *as it's limit.*

(ii) *Show that* $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$.

Solution. *Let* $\epsilon > 0$ *be given. In order to show that* $1/n$ *approaches* 0, *we must show that there exists an integer* $N \in \mathbb{N}$ *such that for all* $n \geq N$,

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \epsilon.$$

But $1/n < \epsilon \Leftrightarrow n > 1/\epsilon$. *Thus, if we choose* $N \in \mathbb{N}$ *such that* $N > 1/\epsilon$, *then for all* $n \geq N$, $1/n < \epsilon$.

(iii) *Consider the sequence* $\{(-1)^{n+1}\}_{n=1}^{\infty}$. *It is intuitively clear that this sequence does not have a limit or it does not approach to any real number. We now prove this by definition. Assume to the contrary, that there exists an* $L \in \mathbb{R}$ *such that the sequence* $\{(-1)^{n+1}\}_{n=1}^{\infty}$ *converges to* L. *Then for* $\epsilon = \frac{1}{2}$, *there exists an* $N \in \mathbb{N}$ *such that*

$$|(-1)^{n+1} - L| < \frac{1}{2},\ \forall\ n \geq N. \tag{1.1}$$

For n *even, (1.1) says*

$$|-1 - L| < \frac{1}{2},\ \forall\ n \geq N. \tag{1.2}$$

while for n *odd, (1.1) says*

$$|1 - L| < \frac{1}{2},\ \forall\ n \geq N. \tag{1.3}$$

which is a contradiction as $2 = |1 + 1| \leq |1 - L| + |1 + L| < 1$.

As a first result we have the following uniqueness theorem:

Theorem 1.0.5. *If $\{a_n\}_1^\infty$ is a sequence and if both $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} a_n = M$ holds, then $L = M$.*

Proof. Suppose that $L \neq M$. Then $|L-M| > 0$. Let $\epsilon = \frac{|L-M|}{2}$. As $\lim_{n\to\infty} a_n = L$, there exists $N_1 \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ for all $n \geq N_1$. Also as $\lim_{n\to\infty} a_n = M$,there exists $N_2 \in \mathbb{N}$ such that $|a_n - M| < \epsilon$ for all $n \geq N_2$. Let $N = \max\{N_1, N_2\}$. Then for all $n \geq N$, $|a_n - L| < \epsilon$ and $|a_n - M| < \epsilon$. Thus $|L-M| \leq |a_n - L| + |a_n - M| < 2\epsilon = |L-M|$, which is a contradiction. ///

Definition 1.0.6. *(***Bounded sequence***): A sequence $\{a_n\}$ is said to be bounded above, if there exists $M \in \mathbb{R}$ such that $a_n \leq M$ for all $n \in \mathbb{N}$. Similarly, we say that a sequence $\{a_n\}$ is bounded below, if there exists $N \in \mathbb{R}$ such that $a_n \geq N$ for all $n \in \mathbb{N}$. Thus a sequence $\{a_n\}$ is said to be bounded if it is both bounded above and below.*

Theorem 1.0.7. *Every convergent sequence is bounded.*

Proof. Let $\{a_n\}$ be a convergent sequence and $L = \lim_{n\to\infty} a_n$. Let $\epsilon = 1$. Then there exists $N \in \mathbb{N}$ such that $|a_n - L| < 1$ for all $n \geq N$. Further,

$$|a_n| = |a_n - L + L| \leq |a_n - L| + |L| < 1 + |L|, \forall\, n \geq N.$$

Let $M = \max\{|a_1|, |a_2|, ..., |a_{n-1}|, 1 + |L|\}$. Then $|a_n| \leq M$ for all $n \in \mathbb{N}$. Hence $\{a_n\}$ is bounded. ///

1.1 Algebra of convergent sequences

Theorem 1.1.1. *Let $\{a_n\}_1^\infty$ and $\{b_n\}_1^\infty$ be two sequences such that $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$. Then*

(i) $\lim_{n\to\infty} (a_n + b_n) = L + M$.

(ii) $\lim_{n\to\infty} (ca_n) = cL$, $c \in \mathbb{R}$..

(iii) $\lim_{n\to\infty} (a_n b_n) = LM$.

(iv) $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{L}{M}$ *if* $M \neq 0$.

Proof. (i) Let $\epsilon > 0$. Since a_n converges to L there exists $N_1 \in \mathbb{N}$ such that

$$|a_n - L| < \epsilon/2 \;\forall\, n \geq N_1.$$

Also, as b_n converges to M there exists $N_2 \in \mathbb{N}$ such that

$$|b_n - M| < \epsilon/2 \;\forall\, n \geq N_2.$$

Thus

$$|(a_n + b_n) - (L + M)| \leq |a_n - L| + |b_N - M| < \epsilon \;\forall\, n \geq N = \max\{N_1.N_2\}.$$

(ii) Easy to prove. Hence left as an exercise to the students.

(iii) Let $\epsilon > 0$. Since a_n is a convergent sequence, it is bounded by M_1 (say). Also as a_n converges to L there exists $N_1 \in \mathbb{N}$ such that

$$|a_n - L| < \epsilon/2M \ \forall \ n \geq N_1.$$

Similarly as b_n converges to M there exists $N_2 \in \mathbb{N}$ such that

$$|b_n - M| < \epsilon/2M_1 \ \forall \ n \geq N_2.$$

Let $N = \max\{N_1, N_2\}$. Then

$$\begin{aligned}|a_n b_n - LM| &= |a_n b_n - a_n M + a_n M - LM| \leq |a_n(b_n - M)| + |M(a_n - L)| \\ &= |a_n||b_n - M| + M|a_n - L| < \epsilon/2 + \epsilon/2 = \epsilon\end{aligned}$$

(iv) In order to prove this, it is enough to prove that if $\lim_{n\to\infty} a_n = L$, $L \neq 0$, then $\lim_{n\to\infty} 1/a_n = 1/L$. Without loss of generality, let us assume that $L > 0$. Let $\epsilon > 0$ be given. As $\{a_n\}$ forms a convergent sequence, it is bounded. Choose $N_1 \in \mathbb{N}$ such that $a_n > L/2$ for all $n \geq N_1$. Also, as a_n converges to L, there exists $N_2 \in \mathbb{N}$ such that $|a_n - L| < L^2\epsilon/2$ for all $n \geq N_2$. Let $N = \max\{N_1, N_2\}$. Then

$$n \geq N \implies \left|\frac{1}{a_n} - \frac{1}{L}\right| = \frac{|a_n - L|}{|a_n L|} < \frac{2}{L^2}\frac{L^2\epsilon}{2} = \epsilon. \qquad ///$$

Examples 1.1.2.

(i) Consider the sequence $\left\{\frac{5}{n^2}\right\}_1^\infty$. *Then* $\lim_{n\to\infty} \frac{5}{n^2} = \lim_{n\to\infty} 5 \cdot \frac{1}{n} \cdot \frac{1}{n} = 5 \cdot \lim_{n\to\infty} \frac{1}{n} \cdot \lim_{n\to\infty} \frac{1}{n} =$ $5 \cdot 0 \cdot 0 = 0$.

(ii) Consider the sequence $\left\{\frac{3n^2 - 6n}{5n^2 + 4}\right\}_1^\infty$. *Notice that* $\frac{3n^2 - 6n}{5n^2 + 4} = \frac{3 - 6/n}{5 + 4/n} \to 3/5$. *Thus*

$$\lim_{n\to\infty} \frac{3n^2 - 6n}{5n^2 + 4} = \lim_{n\to\infty} \frac{3 - 6/n}{5 + 4/n} = \frac{3 - \lim \frac{6}{n}}{5 + \lim_{n\to\infty} \frac{4}{n}} = 3/5.$$