Lecture 20

1 Power series ctd..

The following theorem is very useful in identifying of some Taylor series.

Theorem 1.0.1. *(Term by term differentiation and integration): Suppose* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *converges for* $|x| < R$. *Then*

1. $\sum_{n=0}^{\infty} n a_n x^{n-1}$ *converges in* $|x| < R$ *and is equal to* $f'(x)$.

2. $\sum_{n=0}^{\infty} \frac{1}{n+1} a_n x^{n+1}$ *converges in* $|x| < R$ *and is equal to* $\int f(x)dx$.

From this theorem one concludes that a power series is infinitely differentiable with in its radius of convergence. Now it is natural to ask weather this series coincides with the Taylor series of the resultant function. The answer is yes and it is simple to prove that if $f(x) = \sum a_n x^n$, then $a_n = \frac{f^{(n)}}{n!}$. The above theorem is useful to find Taylor series of some functions.

Example 1.0.2. *The Taylor series of* $f(x) = \tan^{-1} x$ *and a domain of its convergence.*

$$\tan^{-1} x = \int \frac{dx}{1+x^2} = \int 1 - x^2 + x^4 + ...$$
$$= x - \frac{x^3}{3} + \frac{x^5}{5} + ...$$

It is easy to check that the Radius of convergence of this series is equal to 1 *(Try!). At the end point* $x = 1$ *we get interesting sum*

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ... = \tan^{-1}(1) = \frac{\pi}{4}.$$

Though the function $\tan^{-1} x$ *is defined on all of* $\mathbb{R}$, *we see that the power series converges on* $(-1, 1)$. *We can apply Abel's theorem on alternating series to show that the series converges at* $x = 1, -1$.

Proof of Theorem 1.0.1:

Let $f_1(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$, $|x| < R$ and since $f(x)$converges for$|x| < R$, we can choose H such that $|x| + H < R$.Then for$|h| < H$ and$|x| < R$,we have

$$
\begin{aligned}
\left|\frac{f(x+h)-f(x)}{h}-f_1(x)\right| &= \left|\sum_{n=2}^{\infty} a_n \left\{\frac{(x+h)^n - x^n - nhx^{n-1}}{h}\right\}\right| \\
&\le |h| \left|\sum_{n=2}^{\infty} a_n \sum_{k=2}^{n} \binom{n}{k} x^{n-k} h^{k-2}\right| \\
&\le |h| \sum_{n=2}^{\infty} |a_n| \sum_{k=2}^{n} \binom{n}{k} |x|^{n-k} H^{k-2} \\
&= \frac{|h|}{H^2} \sum_{n=2}^{\infty} |a_n| \sum_{k=2}^{n} \binom{n}{k} |x|^{n-k} H^{k} \\
&\le \frac{|h|}{H^2} \sum_{n=0}^{\infty} |a_n| (|x|+H)^n \\
&= \frac{|h|}{H^2} M
\end{aligned}
$$

where $M = \sum_{n=0}^{\infty} |a_n|(|x|+H)^n$ which converges since $|x|+H < R$. Letting h goes to zero, we get

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f_1(x).$$

for$|x| < R$. That is $f_1(x) = f'(x)$.

To prove the term by term integration formula,we define $F(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$. Then from above we see that $F(x) = f(x)$ for all $|x| < R$. Therefore

$$\int_0^x f(t)dt = F(x) - F(0).$$

Hence the proof of the theorem. ///

From this theorem one concludes that a power series is infinitely differentiable with in its radius of convergence.

For approximation, we can use the error approximation of alternating series discussed in the previous section. The total error if we approximate $\tan^{-1} x$ by $s_n(x)$, then the maximum error is $\frac{|x|^{n+1}}{n+1}$.

We note that the power series may converge to a function on small interval, even though the function is defined on a much bigger interval. For example the function $\log(1+x)$ has power series that converges on $(-1,1)$, but $\log(1+x)$ is defined on $(-1,\infty)$. This is obvious due to the fact that the domain of convergence of power series is symmetric about the center. For instance, for a function defined on $(-1,3)$ the radius of convergence

of its power series(about 0) cannot be more than 1.
Another interesting application is to integrate the functions for which we have no "clue". For example,

$$(1)\ erf(x) = \int_0^x e^{-t^2} dt = \int_0^x (1 - \frac{t^2}{1!} + \frac{t^4}{2!} + ... = x - \frac{x^3}{3} + \frac{x^5}{10} + ...$$

$$(2)\ \int_0^x \frac{\sin t}{t} = \int_0^x 1 - \frac{t^2}{3!} + \frac{t^4}{5!} = x - \frac{x^3}{3! \cdot 3} + \frac{x^5}{5! \cdot 5} + ...$$

Example 1.0.3. *Show that* $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + ...$ *for all* $x \in (-1, 1)$.

There are two ways for establishing this. The first one is to show $R_n(x) \to 0$:
It is not difficult to compute the Taylor series around $a = 0$. Also one can easily show that $R_n(x) \to 0$ as $n \to \infty$ for any $x \in (0, 1)$. Fox $x \in (-1, 0)$, we use integration of the series of $f(x) = \frac{1}{1+x}$.

$$\begin{aligned}\log(1+x) &= \int_0^x f(t)dt \\ &= \int_0^x [1 - t + t^2 - t^3 + ...(-1)^n t^n + (-1)^{n+1} \frac{t^n}{1+t}] dt \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} + ... + (-1)^n \frac{t^n}{n+1} + (-1)^{n+1} \int_0^x \frac{t^n}{1+t} dt\end{aligned}$$

Now for any fixed $x \in (-1, 0)$, $f(t) = \frac{1}{1+t}$ is bounded for $t \in (-1, x)$. Hence as $n \to \infty$,

$$|R_n(x)| \leq \left| \int_0^x \frac{t^n}{1+t} dt \right| \leq C \frac{|x|^{n+1}}{n+1}| \leq \frac{C}{n+1} \to 0.$$

The second approach is to use the above theorem to use the seris expansion of $\frac{1}{1+x}$ for $|x| < 1$ to get

$$\log(1+x) = \int_0^x \frac{1}{1+t} dt = \int_0^x [1 - t + t^2 - t^3 + ...] = x - \frac{x^2}{2} + \frac{x^3}{3} + ...$$

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An important application of power series is to solving initial value problems of ODEs:

Example 1.0.4. *Solve* $y'' - 2xy' + y = 0$.

Writing the solution $x(t) = \sum c_n t^n$ and substituting in the equation, we get

$$\sum [(n+1)(n+2)c_{n+2} - (2n-1)c_n] x^n = 0$$

therefore one writes the solution as

$$y(x) = \sum c_{2n}x^{2n} + \sum c_{2n+1}x^{2n+1}$$

All the coefficients are determined using the relation

$$c_{n+2} = \frac{(2n-1)c_n}{(n+1)(n+2)}$$

Both the series in the above converges by ratio test. Indeed,

$$\left|\frac{c_{n+1}}{c_n}\right| = \frac{(2n-1)}{(n+1)(n+2)} \to 0.$$

So it is natural to ask if the series solution always exists for the problem

$$y'' - a_1(x)y' + a_2(x)y = 0. \tag{1.1}$$

The answer is NO. Even if the coefficients a_1, a_2 are infinitely differentiable over whole $\mathbb{R}$, there need not exist a series solution. For example, if $a_1(x) = a_2(x) = \begin{cases} e^{-1/x}, & x \neq 0 \\ 0 & x = 0 \end{cases}$, with $y(0) = 1, y'(0) = 1$. If we compute all derivatives at 0, we get $y^{(n)}(0) = 0$ for all $n \geq 2$. Therefore the Series method yields $y(x) = 1 + x$ which is not a solution of the differential equation in any neighborhood of 0. On the existence of series solutions, the following theorem can be proved.

Theorem 1.0.5. *If a_1, a_2 are real analytic near 0 then series solution exists in a neighborhood of 0 for the problem (1.1).*

Proof of this is beyond the level of this course. So we omit here.