Lecture 21

1 Application of power series

An important application of power series is to solving initial value problems of ODEs:

Example 1.0.1. *Solve* $y'' - 2xy' + y = 0$.

Writing the solution $x(t) = \sum c_n t^n$ and substituting in the equation, we get

$$\sum \left[(n+1)(n+2)c_{n+2} - (2n-1)c_n\right] x^n = 0$$

therefore one writes the solution as

$$y(x) = \sum c_{2n} x^{2n} + \sum c_{2n+1} x^{2n+1}$$

All the coefficients are determined using the relation

$$c_{n+2} = \frac{(2n-1)c_n}{(n+1)(n+2)}$$

Both the series in the above converges by ratio test. Indeed,

$$\left|\frac{c_{n+1}}{c_n}\right| = \frac{(2n-1)}{(n+1)(n+2)} \to 0.$$

So it is natural to ask if the series solution always exists for the problem

$$y'' - a_1(x)y' + a_2(x)y = 0. \tag{1.1}$$

The answer is NO. Even if the coefficients a_1, a_2 are infinitely differentiable over whole $\mathbb{R}$, there need not exist a series solution. For example, if $a_1(x) = a_2(x) = \begin{cases} e^{-1/x}, & x \neq 0 \\ 0 & x = 0 \end{cases}$, with $y(0) = 1, y'(0) = 1$. If we compute all derivatives at 0, we get $y^{(n)}(0) = 0$ for all $n \geq 2$. Therefore the Series method yields $y(x) = 1 + x$ which is not a solution of the differential equation in any neighborhood of 0. On the existence of series solutions, the following theorem can be proved.

Definition 1.0.2. *Two function $u_1(x)$ and $u_2(x)$ are linearly independent if there is no constant C such that $u_1(x) = Cu_2(x)$ for all x.*

Theorem 1.0.3. *If a_1, a_2 are real analytic near 0 then series solution converges in a neighborhood of 0 for the problem (1.1). Moreover there exist two linearly independent solutions.*

Proof of this is beyond the level of this course. So we omit here.