Lecture 29

1 Definite Integral

1.1 Definition, Necessary & sufficient conditions

Let $f : [a, b] \to \mathbb{R}$ be a bounded real valued function on the closed, bounded interval $[a, b]$. Also let m, M be the infimum and supremum of $f(x)$ on $[a, b]$, respectively.

Definition 1.1.1. *A partition P of $[a, b]$ is an ordered set $P = \{a = x_0, x_1, x_2, ..., x_n = b\}$ such that $x_0 < x_1 < ... < x_n$.*

Let m_k and M_k be the infimum and supremum of $f(x)$ on the subinterval $[x_{k-1}, x_k]$, respectively.

Definition 1.1.2. ***Lower sum:*** *The Lower sum, denoted with $L(P, f)$ of $f(x)$ with respect to the partition P is given by*

$$L(P, f) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}).$$

Definition 1.1.3. ***Upper sum:*** *The Upper sum, denoted with $U(P, f)$ of $f(x)$ with respect to the partition P is given by*

$$U(P, f) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}).$$

For a given partition P, $U(P, f) \geq L(P, f)$. In fact the same inequality holds for any two partitions. (see Lemma (1.1.6) below.)

Definition 1.1.4. ***Refinement of a Partition:*** *A partition Q is called a refinement of the partition P if $P \subseteq Q$.*

The following is a simple observation.

Lemma 1.1.5. *If Q is a refinement of P, then*

$$L(P, f) \leq L(Q, f) \text{ and } U(P, f) \geq U(Q, f).$$

Proof. Let $P = \{x_0, x_1, x_2, ..., x_{k-1}, x_k, ..., x_n\}$ and $Q = \{x_0, x_1, x_2, ..., x_{k-1}, z, x_k, ..., x_n\}$. Then

$$\begin{aligned} L(P, f) &= m_0(x_1 - x_0) + ... + m_k(x_k - x_{k-1}) + ... + m_{n-1}(x_n - x_{n-1}) \\ &\leq m_0(x_1 - x_0) + ... + m_k^{'}(x_k - z) + m_k^{''}(z - x_{k-1}) + ... + m_{n-1}(x_n - x_{n-1}) \\ &= L(Q, f) \end{aligned}$$

where $m_k^{'} = \inf\limits_{[z, x_k]} f(x)$ and $m_k^{''} = \inf\limits_{[x_{k-1}, z]} f(x)$.

Lemma 1.1.6. *If P_1 and P_2 be any two partitions, then*

$$L(P_1, f) \leq U(P_2, f).$$

Proof. Let $Q = P_1 \cup P_2$. Then Q is a refinement of both P_1 and P_2. So by Lemma (1.1.8),

$$L(P_1, f) \leq L(Q, f) \leq U(Q, f) \leq U(P_2, f).$$

Definition 1.1.7. *Let $\mathcal{P}$ be the collection of all possible partitions of $[a, b]$. Then the upper integral of f is*

$$\int_a^{\bar{b}} f = \inf\{U(P, f) : P \in \mathcal{P}\}$$

and lower integral of f is

$$\int_{\underline{a}}^b f = \sup\{L(P, f) : P \in \mathcal{P}\}.$$

An immediate consequence of Lemma (1.1.6) is

Lemma 1.1.8. *For a bounded function $f : [a, b] \to I\!R$,*

$$\int_{\underline{a}}^b f \leq \int_a^{\bar{b}} f.$$

Definition 1.1.9. ***Riemann integrability:*** *$f : [a, b] \to \mathbb{R}$ is said to be Riemann integrable if*

$$\int_{\underline{a}}^b f = \int_a^{\bar{b}} f$$

and the value of the limit is denoted with $\displaystyle\int_a^b f(x)dx$. We say $f \in \mathcal{R}[a, b]$.

Example 1.1.10. *$f(x) = x$ on $[0, 1]$*

Consider the sequence of partitions $P_n = \{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}, \frac{n}{n}\}$. Then

$$\begin{aligned} L(P_n, f) &= 0 \cdot \frac{1}{n} + \frac{1}{n} \cdot \frac{1}{n} + ... + \frac{n-1}{n}\frac{1}{n} \\ &= \frac{1}{n^2}[1 + 2 + ... + (n-1)] \\ &= \frac{n(n-1)}{2n^2} \end{aligned}$$

Thus $\lim_{n\to\infty} L(P_n, f) = \dfrac{1}{2}$. Hence from the definition $\underline{\int_0^1} f(x)dx \geq \frac{1}{2}$. Similarly

$$\begin{aligned} U(P_n, f) &= \frac{1}{n}\cdot\frac{1}{n} + \frac{2}{n}\cdot\frac{1}{n} + ... + \frac{n}{n}\frac{1}{n} \\ &= \frac{1}{n^2}[1+2+...+n] \\ &= \frac{n(n+1)}{2n^2} \end{aligned}$$

Hence $\lim_{n\to\infty} U(P_n, f) = \dfrac{1}{2}$. Again from the definition $\overline{\int_0^1} f(x)dx \leq \frac{1}{2}$.

Example 1.1.11. *$f(x) = x^2$ on $[0, 1]$*

Consider the sequence of partitions $P_n = \{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}, \frac{n}{n}\}$. Then

$$\begin{aligned} U(P_n, f) &= \frac{1}{n^2}\cdot\frac{1}{n} + \left(\frac{2}{n}\right)^2\cdot\frac{1}{n} + ... + \left(\frac{n}{n}\right)^2\frac{1}{n} \\ &= \frac{1}{n^3}[1+2^2+...+n^2] \\ &= \frac{n(n+1)(2n+1)}{6n^3} \end{aligned}$$

Thus $\lim_{n\to\infty} U(P_n, f) = \dfrac{1}{3}$. Similarly

$$\begin{aligned} L(P_n, f) &= 0\cdot\frac{1}{n} + \left(\frac{1}{n}\right)^2\cdot\frac{1}{n} + ... + \left(\frac{n-1}{n}\right)^2\frac{1}{n} \\ &= \frac{1}{n^3}[1+2^2+...+(n-1)^2] \\ &= \frac{n(n-1)(2n-1)}{6n^3} \end{aligned}$$

Therefore, $\lim_{n\to\infty} L(P_n, f) = \dfrac{1}{3}$.
Hence from the definition $\underline{\int_a^b} f \geq 1/3$ and $\overline{\int_a^b} f \leq 1/3$.

Remark 1.1. *In the above two examples $\underline{\int_0^1} f = \overline{\int_0^1} f$ thanks to Lemma 1.1.8*

The following example illustrates the non-integrability.

Example 1.1.12. *On $[0, 1]$, consider the function* $f(x) = \begin{cases} 1, & x \in Q, \\ 0, & x \notin Q. \end{cases}$

Let P be a partition of $[0,1]$. In any sub interval $[x_{k-1}, x_k]$, there exists a rational number and irrational number. Then the supremum in any subinterval is 1 and infimum is 0. Therefore, $L(P,f) = 0$ and $U(P,f) = 1$. Hence $\underline{\int_0^1} f \neq \overline{\int_0^1} f$.

Necessary and sufficient condition for integrability

Theorem 1.1.13. *A bounded function $f \in \mathcal{R}[a,b]$ if and only if for every $\epsilon > 0$, there exists a partition P_ϵ such that*

$$U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon.$$

Proof. $\Leftarrow$: Let $\epsilon > 0$. Then from the definition of upper and lower integral we have

$$\overline{\int_a^b} f - \underline{\int_a^b} f \leq U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon(\text{ by hypothesis}).$$

Thus the conclusion follows as $\epsilon > 0$ is arbitrary.
$\Rightarrow$: Conversely, since $\overline{\int_a^b} f$ is the infimum, for any $\epsilon > 0$, there exists a partition P_1 such that

$$U(P_1, f) < \overline{\int_a^b} f + \frac{\epsilon}{2}.$$

Similarly there exists a partition P_2 such that

$$L(P_2, f) > \underline{\int_a^b} f - \frac{\epsilon}{2}.$$

Let $P_\epsilon = P_1 \cup P_2$. Then P_ϵ is a refinement of P_1 and P_2. Hence

$$\begin{aligned} U(P_\epsilon, f) - L(P_\epsilon, f) &\leq U(P_1, f) - L(P_2, f) \\ &\leq \overline{\int_a^b} f + \frac{\epsilon}{2} - \underline{\int_a^b} f + \frac{\epsilon}{2} \\ &= \epsilon \text{ (as } f \text{ is integrable, } \overline{\int_a^b} f = \underline{\int_a^b} f) \end{aligned}$$

This complete the theorem. ///

Now it is easy to see that the functions considered in Example 1 and Example 2 are integrable. For any $\epsilon > 0$, we can find n (large) and P_n such that $\frac{1}{n} < \epsilon$. Then

$$U(P_n, f) - L(P_n, f) = \frac{1}{2n^2}(n(n+1) - n(n-1)) = \frac{1}{n} < \epsilon.$$

As a consequence we have the following sequential characterization:

Theorem 1.1.14. *$f : [a,b] \to \mathbb{R}$ is integrable if and only if there exists a sequence $\{P_n\}$ of partitions of $[a,b]$ such that*

$$\lim_{n\to\infty} U(P_n, f) - L(P_n, f) = 0.$$

Example 1.1.15. *Consider $f(x) = \dfrac{1}{x}$ on $[1, b]$.*

Divide the interval in geometric progression and compute $U(P_n, f)$ and $L(P_n, f)$ to show that $f \in \mathcal{R}[1, b]$.

Solution: Let $P_n = \{1, r, r^2, ..., r^n = b\}$ be a partition on $[1, b]$. Then

$$\begin{aligned}
U(P_n, f) &= f(1)(r-1) + f(r)(r^2 - r) + ... + f(r^{n-1})(r^n - r^{n-1}) \\
&= (r-1) + \frac{1}{r} r(r-1) + .. + \frac{1}{r^{n-1}} r^{n-1}(r-1) \\
&= n(r-1) \\
&= n(b^{\frac{1}{n}} - 1)
\end{aligned}$$

Therefore $\lim\limits_{n\to\infty} U(P_n, f) = \lim\limits_{n\to\infty} \dfrac{b^{\frac{1}{n}} - 1}{\frac{1}{n}} = \lim\limits_{n\to\infty} \dfrac{b^{\frac{1}{n}} \ln b (\frac{-1}{n^2})}{\frac{-1}{n^2}} = \ln b.$

Similarly

$$\begin{aligned}
L(P_n, f) &= f(r)(r-1) + f(r^2)(r^2 - r) + ... + f(r^n)(r^n - r^{n-1}) \\
&= \frac{1}{r}(r-1) + .. + \frac{1}{r^n} r^{n-1}(r-1) \\
&= \frac{n}{r}(b^{\frac{1}{n}} - 1) \\
&= n(1 - \frac{1}{b^{1/n}}) \\
&= \frac{b^{1/n} - 1}{b^{1/n} \frac{1}{n}} \to \ln b \text{ as } n \to \infty.
\end{aligned}$$