Lecture 3

1 Sequences ctd..

Theorem 1.0.1 (Sandwich theorem for sequences)**.** *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three sequences such that $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$. If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.*

Proof. Let $\epsilon > 0$ be given. As $\lim_{n\to\infty} a_n = L$, there exists $N_1 \in \mathbb{N}$ such that

$$n \geq N_1 \implies |a_n - L| < \epsilon. \tag{1.1}$$

Similarly as $\lim_{n\to\infty} c_n = L$, there exists $N_2 \in \mathbb{N}$

$$n \geq N_2 \implies |c_n - L| < \epsilon. \tag{1.2}$$

Let $N = \max\{N_1, N_2\}$. Then, $L - \epsilon < a_n$ (from (1.1)) and $c_n \leq L + \epsilon$ (from (1.2)). Thus

$$L - \epsilon < a_n \leq b_n \leq c_n \leq L + \epsilon.$$

Thus $|b_n - L| < \epsilon$ for all $n \geq N$. Hence the proof. ///

Examples 1.0.2.

(i) *Consider the sequence* $\left\{\frac{\cos n}{n}\right\}_{n=1}^{\infty}$. *Then* $\frac{-1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n}$. *Hence by Sandwich theorem* $\lim_{n\to\infty} \frac{\cos n}{n} = 0$.

(ii) *As* $0 \leq \frac{1}{2^n} \leq \frac{1}{n}$ *and* $\frac{1}{n} \to 0$ *as* $n \to \infty$, $\frac{1}{2^n}$ *also converges to* 0 *by Sandwich theorem.*

(iv) *If* $b > 0$, *then* $\lim_{n\to\infty} \sqrt[n]{b} = 1$.

First assume that $b > 1$. *Let* $a_n = b^{\frac{1}{n}} - 1$. *As* $b > 1$, $a_n > 0$ *for all* $n \in \mathbb{N}$. *Further,*

$$b = (1 + a_n)^n \geq 1 + na_n.$$

Then $0 \leq a_n \leq \frac{b-1}{n}$. *Thus* $a_n \to 0$, *i.e.,* $b^{\frac{1}{n}} \to 1$ *as* $n \to \infty$.
Now if $b < 1$, *then take* $c = \frac{1}{b}$ *and it is easy to show the result.* ///

Examples 1.0.3.

(i) $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

(ii) *If* $x > 0$ *then* $\lim_{n\to\infty} \frac{n^x}{(1+x)^n} = 0$.

(iii) *If* $p > 0$, *then* $\lim_{n\to\infty} \frac{\log(n)}{n^p} = 0$.

Solution. (i) Let $a_n = n^{\frac{1}{n}} - 1$. Then $0 \leq a_n \leq 1$ for all $n \in \mathbb{N}$. Further,

$$n = (1+a_n)^n \geq \frac{n(n-1)}{2} a_n^2.$$

Thus $0 \leq a_n \leq \sqrt{\frac{2}{(n-1)}}$ $(n \geq 2)$. As $\sqrt{\frac{2}{(n-1)}} \to 0$ as $n \to \infty$, by Sandwich theorem, $a_n \to 0$, i.e., $n^{\frac{1}{n}} \to 1$ as $n \to \infty$.

(ii) Let k be an integer such that $k > x,\ k > 0$. Then for $n > 2k$,

$$(1+x)^n > \ {}_nC_k x^k = \frac{n!}{k!(n-k)!} x^k = \frac{x^k}{k!} \prod_{i=1}^{k} [n-i+1] > \frac{n^k}{2^k} \frac{x^k}{k!}.$$

Hence,

$$0 < \frac{n^x}{(1+x)^n} < \frac{2^k k!}{x^k} n^{x-k}.$$

As $x - k < 0$, $n^{x-k} \to 0$. Thus $\dfrac{n^x}{(1+x)^n} \to 0$ as $n \to \infty$.

(iii) By Archimedian property, for any $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that

$$m \leq n^p < (m+1)$$

or equivalently

$$m^{\frac{1}{p}} \leq n < (m+1)^{\frac{1}{p}}.$$

Therefore,

$$\frac{1}{n^p} < \frac{\log n}{n^p} < \frac{1}{p} \frac{\log(m+1)}{m}.$$

Also

$$\frac{1}{p} \frac{\log(m+1)}{m} = \frac{1}{p} \frac{\log(m+1)}{m+1} \frac{m+1}{m}.$$

So it is enough to show $\frac{\log n}{n} \to 0$ as $n \to \infty$. for this,
let us start with $\epsilon > 0$. From the previous problem we have $n^{\frac{1}{n}} \to 1$ as $n \to \infty$. This implies there exists $N \in \mathbb{N}$ such that

$$n^{\frac{1}{n}} \in (e^{-\epsilon}, e^{\epsilon}),\ \forall\ n \geq N \text{ (equivalently) } \frac{\log n}{n} \in (-\epsilon, \epsilon),\ \forall\ n \geq N.$$

That is, $\frac{\log n}{n} \to 0$. ///

Definition 1.0.4. Subsequence: *Let $\{a_n\}$ be a sequence and $\{n_1, n_2, ...\}$ be a sequence of positive integers such that $i > j$ implies $n_i > n_j$. Then the sequence $\{a_{n_i}\}_{i=1}^{\infty}$ is called a subsequence of $\{a_n\}$.*

Theorem 1.0.5. *If the sequence of real numbers $\{a_n\}_1^{\infty}$, is convergent to L, then any subsequence of $\{a_n\}$ is also convergent to L.*

Proof. Let $\{n_i\}_{i=1}^{\infty}$ be a sequence of positive integers such that $\{a_{n_i}\}_{i=1}^{\infty}$ forms a subsequence of

$\{a_n\}$. Let $\epsilon > 0$ be given. As $\{a_n\}$ converges to L, there exists $N \in \mathbb{N}$ such that

$$|a_n - L| < \epsilon, \ \forall \ n \geq N.$$

Choose $M \in \mathbb{N}$ such that $n_i \geq N$ for $i \geq M$. Then

$$|a_{n_i} - L| < \epsilon, \ \forall \ i \geq M.$$

Hence the proof. ///

Theorem 1.0.6. *1. If a sequence $\{a_n\}$ converges to a. Then $\{|a_n|\}$ converges to $|a|$.*

2. If $a_n \geq 0$ and $a_n \to a$, then $\left\{\sqrt{a_n}\right\}$ converges to $\sqrt{a}$.

Proof. Proof of (1) follows from the inequality

$$||a| - |b|| \leq |a - b|, \ \ \forall a, b \in \mathbb{R}.$$

(2) follows from the fact that if $a \neq 0$, then

$$|\sqrt{a_n} - \sqrt{a}| \leq \frac{|a_n - a|}{|\sqrt{a_n} + \sqrt{a}|}.$$

The case $a = 0$ is easy and is left as an exercise.

Theorem 1.0.7. *If a sequence $\{a_n\}$ converges to a, then $\left\{\frac{1}{n}\sum_{k=1}^{n} a_k\right\}$ also converges to a.*

Proof. Proof is left as an exercise. ///

Next we study some fine properties of sequences that imply convergence/divergence.

Theorem 1.0.8. *For any sequence $\{a_n\}$ with $a_n > 0$*

$$\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

provided the limit on the right side exists.

Proof. Let $\epsilon > 0$ be arbitrary. Suppose the second limit exists (say l), then there exists $N \in \mathbb{N}$ such that

$$l - \epsilon < \frac{a_{n+1}}{a_n} < l + \epsilon, \ \ \forall n \geq N.$$

Taking $n = N, N+1, ..., m-1$ and multiplying we get

$$(l - \epsilon)^{m-N} < \frac{a_m}{a_N} < (l + \epsilon)^{m-N}, \ \ \forall m \geq N + 1$$

equivalently,

$$(l - \epsilon)^{1-\frac{N}{m}} a_N^{\frac{1}{m}} < (a_m)^{\frac{1}{m}} < (l + \epsilon)^{1-\frac{N}{m}} a_N^{\frac{1}{m}}, \ \ \forall m \geq N + 1.$$

Now the result follows from the fact that $\lim_{m\to\infty}(l\pm\epsilon)^{1-N/m}a_N^{1/m} = l\pm\epsilon$. ///

Corollary 1.0.9. *(i) If* $a_n > 0$ *and* $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = l < 1$*, then* $\lim_{n\to\infty} a_n = 0$.

(ii) If $a_n > 0$ *and* $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = l > 1$*, then* $a_n \to \infty$.

Proof. If $L < 1$, then choose ϵ_0 such that $L + \epsilon_0 < 1$. Then for this ϵ_0 there exists N_0 such that

$$n \geq N_0 \implies \frac{a_{n+1}}{a_n} < L + \epsilon_0$$

Therefore from the hypothesis of (i)

$$n \geq N_0 \implies a_n^{\frac{1}{n}} < L + \epsilon_0$$

Therefore $a_n < (L+\epsilon_0)^n \to 0$ as $n \to \infty$. A similar argument will imply the (ii). ///

Examples 1.0.10. (i) $\lim a^{1/n} = 1$, *if* $a > 0$ (ii) $\lim n^\alpha x^n = 0$, *if* $|x| < 1$ *and* $\alpha \in \mathbb{R}$.
Solution: (i) *Take* $a_n = a$*, then* $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = 1$.
(ii) *If* $x \neq 0$*, take* $a_n = n^\alpha x^n$*, then* $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}(1+\frac{1}{n})^\alpha |x| = |x|$.