Lecture 30

Definition 1.0.1. *Riemann sum: For a partition $P = \{x_1, x_2, \cdots, x_n\}$, the Riemann sum $S(P, f)$ is defined as* $S(P, f) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}),\ \xi_i \in [x_{i-1}, x_i]$.

Then it is easy to show the following

Remark 1.0.1. *If* $m = \inf_{[a,b]} f(x)$, *and* $M = \sup_{[a,b]} f(x)$. *Then*

$$m(b-a) \le L(P, f \le S(P, f) \le U(P, f) \le M(b-a).$$

In fact, one has the following Darboux theorem:

Theorem 1.0.2. *Let f $f : [a, b] \to I\!R$ be a Riemann integrable function. Then for a given $\epsilon > 0$, there exists $\delta > 0$ such that for any partition P with $\|P\| := \max_{1 \le i \le n} |x_i - x_{i-1}| < \delta$, we have*

$$|S(P, f) - \int_a^b f(x)dx| < \epsilon.$$

Corollary 1.0.3. *If $f \in \mathcal{R}[a, b]$, then for any sequence of partitions $\{P_n\}$ with $\|P_n\| \to 0$, we have $L(P_n, f) \to \int_a^b f(x)dx$ and $U(P_n, f) \to \int_a^b f(x)dx$.*

Remark 1.0.2. *From the above theorem, we note that if there exists a sequence of partition $\{P_n\}$ such that $\|P_n\| \to 0$ and $U(P_n, f) - L(P_n, f) \not\to 0$ as $n \to \infty$, then f is not integrable.*

Problem 1.0.1. *Show that the function $f : [0, 1] \to I\!R$*

$$f(x) = \begin{cases} 1 + x & x \in \mathbb{Q} \\ 1 - x & x \notin \mathbb{Q} \end{cases}$$

is not integrable.

Solution: Consider the sequence of partitions $P_n = \{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n}{n} = 1\}$. Then

$$\begin{aligned} U(P_n, f) &= (1 + \frac{1}{n})\frac{1}{n} + (1 + \frac{2}{n})\frac{1}{n} + ... + (1 + \frac{n}{n})\frac{1}{n} \\ &= 1 + \frac{1}{n^2}(1 + 2 + ... + n) \\ &\to \frac{3}{2} \text{ as } n \to \infty \end{aligned}$$

Now using the fact that infimum of f on $[0, \frac{1}{n}]$ is $1 - \frac{1}{n}$, though it is not achieved, we get

$$L(P_n, f) = (1 - \frac{1}{n})\frac{1}{n} + (1 - \frac{2}{n})\frac{1}{n} + ... + (1 - \frac{n}{n})\frac{1}{n} \to \frac{1}{2} \text{ as } n \to \infty.$$

Hence f is not integrable.

Theorem 1.0.4. *Suppose f is a continuous function on $[a, b]$. Then $f \in \mathcal{R}[a, b]$.*

Proof. Let $\epsilon > 0$. By the Theorem on neccessary and sufficient condition for integrability , we need to show the existence of a partition P such that

$$U(P, f) - L(P, f) < \epsilon.$$

Since f is continuous on $[a, b]$, this implies f is uniformly continuous on $[a, b]$. Therefore there exists $\delta > 0$ such that

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\epsilon}{(b-a)}. \tag{1.1}$$

Now choose a partition P such that

$$\sup_{1 \leq k \leq n} |x_k - x_{k-1}| < \delta. \tag{1.2}$$

As f is continuous on $[a, b]$ there exist $x_k', x_k'' \in (x_{k-1}, x_k)$ such that $m_k = f(x_k')$ and $M_k = f(x_k'')$. By (1.2), $|x_k' - x_k''| < \delta$ and hence by (1.1) $|f(x_k'') - f(x_k')| < \frac{\epsilon}{(b-a)}$. Thus

$$\begin{aligned} U(P, f - L(P, f) &= \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1}) \\ &= \sum_{k=1}^{n} (f(x_k'') - f(x_k'))(x_k - x_{k-1}) \\ &\leq \frac{\epsilon}{(b-a)} \sum_{k=1}^{n} (x_k - x_{k-1}) = \frac{\epsilon}{(b-a)}(b-a) = \epsilon. \end{aligned}$$

Therefore $f \in \mathcal{R}[a, b]$.

Integrability and discontinuous functions:
We study the effect of discontinuity on integrability of a function $f(x)$.

Example 1.0.5. *Consider the following function $f : [0, 1] \to \mathbb{R}$.*

$$f(x) = \begin{cases} 1, & x \neq \frac{1}{2} \\ 0, & x = \frac{1}{2} \end{cases}$$

Clearly $U(P, f) = 1$ for any partition P. We notice that $L(P, f)$ will be less than 1. We can try to isolate the point $x = \frac{1}{2}$ in a subinterval of small length. Consider the partition $P_\epsilon = \{0, \frac{1}{2} - \frac{\epsilon}{2}, \frac{1}{2} + \frac{\epsilon}{2}, 1\}$. Then $L(P_\epsilon, f) = (\dfrac{1}{2} - \dfrac{\epsilon}{2}) + (1 - \dfrac{1}{2} - \dfrac{\epsilon}{2}) = 1 - \epsilon$. Therefore, for given $\epsilon > 0$ we have $U(P_\epsilon, f) - L(P_\epsilon, f) = \epsilon$. Hence f is integrable.

In fact we have the following theorem.

Theorem 1.0.6. *Suppose $f : [a, b] \to \mathbb{R}$ be a bounded function which has finitely many discontinuities. Then $f \in \mathcal{R}[a, b]$.*

Proof follows by constructing suitable partition with sub-intervals of sufficiently small length around the discontinuities as observed in the above example.Next we have the following theorem

Theorem 1.0.7. *Let f be a monotonically decreasing function on $[a, b]$, then f is integrable.*

Proof. Let $P_n = \{x_0, x_1, \cdots, x_n\}$ be a partition of $[a, b]$ with $x_k - x_{k-1} = \frac{b-a}{n}$. Since f is monotone, it is bounded (exercise!). Also $m_k = \inf_{[x_{k-1}, x_k]} f(x) = f(x_k)$ and $M_k = \sup_{[x_{k-1}, x_k]} f(x) = f(x_{k-1})$. Then

$$U(P_n, f) - L(P_n, f) = \sum_{k=1}^{n} [f(x_{k-1}) - f(x_k)] \frac{b-a}{n} = \frac{1}{n}(b-a)(f(b) - f(a)) \to 0$$

as $n \to \infty$. Therefore by the sequential characterization of integrability, $f \in \mathcal{R}[a, b]$. ///

1.1 Properties of Definite Integral:

Property 1: For a constant $c \in \mathbb{R}$, $\int_a^b cf(x)dx = c \int_a^b f(x)dx$.
Property 2: Let $f_1, f_2 \in \mathcal{R}[a, b]$. Then

$$\int_a^b (f_1 + f_2)(x)dx = \int_a^b f_1(x)dx + \int_a^b f_2(x)dx.$$

Easy to show that for any partition P,

$$U(P, f_1 + f_2) \leq U(P, f_1) + U(P, f_2) \tag{1.3}$$
$$L(P, f_1 + f_2) \geq L(P, f_1) + L(P, f_2) \tag{1.4}$$

Since f_1, f_2 are integrable, for $\epsilon > 0$ there exists P_1, P_2 such that

$$U(P_1, f_1) - L(P_1, f_1) < \epsilon$$
$$U(P_2, f_2) - L(P_2, f_2) < \epsilon$$

Now taking $P = P_1 \cup P_2$, if necessary, we assume

$$U(P, f_1) - L(P, f_1) < \epsilon, \quad U(P, f_1) - L(P, f_2) < \epsilon \tag{1.5}$$

Therefore, using (1.3)-(1.5) we get

$$\begin{aligned} U(P, f_1+f_2) - L(P, f_1+f_2) &\le U(P, f_1) + U(P, f_2) - L(P, f_2) - L(P, f_2) \\ &< \epsilon + \epsilon = 2\epsilon. \end{aligned}$$

Hence, $f_1 + f_2$ is integrable.

$$\begin{aligned} \int_a^b (f_1+f_2)(x)dx = \lim_{n\to\infty} S(P_n, f_1+f_2) &= \lim_{n\to\infty} \sum_{k=1}^{n} (f_1+f_2)(\xi_k)(x_k - x_{k-1}) \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} f_1(\xi_k)(x_k - x_{k-1}) + \lim_{n\to\infty} \sum_{k=1}^{n} f_2(\xi_k)(x_k - x_{k-1}) \\ &= \int_a^b f_1(x)dx + \int_a^b f_2(x)dx \end{aligned}$$

Property 3: If $f(x) \le g(x)$ on $[a, b]$. Then

$$\int_a^b f(x)dx \le \int_a^b g(x)dx.$$

First we note that $m \le f(x) \le M$ implies $m(b-a) \le \int_a^b f(x) \le M(b-a)$. Then Property 1 and $f(x) \le g(x)$ imply $\int_a^b (g-f) \ge 0$ or $\int_a^b g(x)dx \ge \int_a^b f(x)dx$.

Property 4: If $f \in \mathcal{R}[a, b]$ then $|f| \in \mathcal{R}[a, b]$ and $|\int_a^b f(x)dx| \le \int_a^b |f|(x)dx$. Let $m_k^{'} = \inf_{[x_{k-1}, x_k]} |f|(x)$ and $M_k^{'} = \sup_{[x_{k-1}, x_k]} |f|(x)$. Then we claim

Claim: $M_k - m_k \ge M_k^{'} - m_k^{'}$
Proof of Claim: Note that for $x, y \in [x_{i-1}, x_i]$,

$$|f|(x) - |f|(y) \le |f(x) - f(y)| \le M_i(f) - m_i(f).$$

Now take supremum over x and infimum over y, to conclude the claim.

Now since f is integrable, there exists partitions $\{P_n\}$ such that $\lim_{n\to\infty} U(P_n, f) - L(P_n, f) = 0$. i.e.,

$$\lim_{n\to\infty} \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1}) = 0.$$

This implies

$$\lim_{n\to\infty}\sum_{k=1}^{n}(M_k^{'} - m_k^{'})(x_k - x_{k-1}) = 0.$$

Hence $|f|$ is integrable. Note that $-|f| \leq f \leq |f|$. Thus by Property 3 we get

$$-\int_a^b |f|(x)dx \leq \int_a^b f(x)dx \leq \int_a^b |f|(x)dx \Longrightarrow |\int_a^b f(x)dx| \leq \int_a^b |f|(x)dx.$$