Lecture 31

Domain decompsition property

Let f be bounded on $[a, b]$ and let $c \in (a, b)$. Then f is integrable on $[a, b]$ if and only if f is integrable on $[a, c]$ and $[c, b]$. In this cases

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Let f be integrable on $[a, b]$. For $\epsilon > 0$, there exists partition P such that

$$U(P, f) - L(P, f) < \epsilon. \tag{1.1}$$

With out loss of generality we can assume that P contain c. (otherwise we can refine P by adding c and the difference will be closer than before) Let $P_1 = P \cap [a, c]$ and $P_2 = P \cap [c, b]$. Then P_1 and P_2 are partitions on $[a, c]$ and $[c, b]$ respectively. Also by (1.1), $U(P_1, f) - L(P_1, f) < \epsilon$ and $U(P_2, f) - L(P_2, f) < \epsilon$. This implies f is integrable on $[a, c]$ and $[c, b]$. Conversely, suppose f is integrable on $[a, c]$ and $[c, b]$. Then for $\epsilon > 0$, there exists partitions P_1 of $[a, c]$ and P_2 of $[c, b]$ such that $U(P_1, f) - L(P_1, f) < \frac{\epsilon}{2}$ and $U(P_2, f) - L(P_2, f) < \frac{\epsilon}{2}$. Now take $P = P_1 \cup P_2$. Then $U(P, f) - L(P, f) < \epsilon$. So by Remark **??**, there exists $\{P_n\}$ such that

$$\begin{aligned}
\int_a^b f(x)dx &= \lim_{n\to\infty} S(P_n, f) = \lim_{n\to\infty} \sum_{P_n} f(\xi_k)(x_{k-1} - x_k) \\
&= \sum_{P_n \cap [a,c]} f(\xi_k)(x_k - x_{k-1}) + \sum_{P_n \cap [c,b]} f(\xi_k)(x_k - x_{k-1}) \\
&\to \int_a^c f(x)dx + \int_c^b f(x)dx
\end{aligned}$$

Example 1.0.1. *Consider the following function $f : [0, 1] \to \mathbb{R}$.*

$$f(x) = \begin{cases} 1 & x = \frac{1}{n} \text{ for some } n \in \mathbb{N}, n \geq 2 \\ 0 & x \neq \frac{1}{n} \end{cases}$$

Solution: Let $\epsilon > 0$. Choose N such that $\dfrac{1}{N} < \dfrac{\epsilon}{2}$. Note that $f(x)$ has only finitely many discontinuities in $[\frac{1}{N}, 1]$ say $\xi_1, \xi_2, ..., \xi_r$. Define the partition P_ϵ as

$$P_\epsilon = \{0, \frac{1}{N}, \xi_1 - \frac{\epsilon}{4r}, \xi_1 + \frac{\epsilon}{4r}, ..., \xi_r - \frac{\epsilon}{4r}, \xi_r + \frac{\epsilon}{4r}, 1\}.$$

Since ξ_r is the last discontinuity, $f = 0$ in $[\xi_r + \frac{\epsilon}{4r}, 1]$. Now $L(P_\epsilon, f) = 0$ and

$$\begin{aligned} U(P_\epsilon, f) &= 1 \cdot \frac{1}{N} + \frac{\epsilon}{2r} + \frac{\epsilon}{2r} + ... + \frac{\epsilon}{2r} + 0 \cdot (1 - \xi_r - \frac{\epsilon}{4r}) \\ &= \frac{1}{N} + \frac{\epsilon}{2} < \epsilon. \end{aligned}$$

Therefore f is Riemann integrable. ///

Example 1.0.2. *Consider the following function $f : [0, 1] \to \mathbb{R}$.*

$$f(x) = \begin{cases} 0 & x \in Q \\ \sin \frac{1}{x} & x \notin Q \end{cases}$$

Solution: We will show that f is not integrable on a sub interval of $[0, 1]$. Consider the f on the subinterval $I_1 = [\frac{2}{\pi}, 1]$. Clearly $L(P, f) = 0$ for any partition P of I_1 because $f(x) \geq 0$ in the sub interval $[\frac{2}{\pi}, 1]$. Let M_k be the Supremum of f on subintervals $[x_{k-1}, x_k]$ of $[\frac{2}{\pi}, 1]$. Also the minimum of $M_k's$ is $\sin 1$. Therefore,

$$U(P, f) = \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) > (1 - \frac{2}{\pi}) \sin 1.$$

Hence $U(P, f) - L(P, f)$ can not be made less than ϵ for any $\epsilon < (1 - \frac{2}{\pi}) \sin 1$. Therefore f is not Riemann integrable. ///

The discussion above suggests that if a function has countably many discontinuities then it is integrable. In fact, we have the following:

Definition 1.0.3. *(Zero set)*
A subset $A \subset \mathbb{R}$ is said to be zero set if for every $\epsilon > 0$, there exists countable number of intervals I_i such that $A \subset \cup_i I_i$ and $\sum_i |I_i| < \epsilon$.

So it is clear that any finite set is a zero set. Moreover any countable set can always be covered by intervals of length $\epsilon/2^i$ for $i \in \mathbb{N}$. Moreover any countable union of zero sets is also a zero set.

Theorem 1.0.4. *(Riemann-Lebesgue theorem)*
A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if its set of discontinuous points is a zero set.

Mean Value Theorem

Theorem 1.0.5. *Let $f(x)$ be a continuous function on $[a, b]$. Then there exists $\xi \in [a, b]$ such that*

$$\int_a^b f(x)dx = f(\xi)(b - a).$$

Proof. Let $m = \min_{x \in [a,b]} f(x)$ and $M = \max_{x \in [a,b]} f(x)$. Then by Property 3, we have

$$m(b-a) \leq \int_a^b f \leq M(b-a),$$

i.e.

$$m \leq \frac{1}{(b-a)} \int_a^b f \leq M.$$

Now since $f(x)$ is continuous, it attains all values between it's maximum and minimum values. Therefore there exists $\xi \in [a,b]$ such that $f(\xi) = \frac{1}{(b-a)} \int_a^b f$. ///

Theorem 1.0.6. ***Fundamental Theorem:*** *Let $f(x)$ be a continuous function on $[a,b]$ and let $\phi(x) = \int_a^x f(s)ds$. Then ϕ is differentiable and $\phi'(x) = f(x)$.*

Proof. As $\frac{\phi(x+\Delta x) - \phi(x)}{\Delta x} = \frac{1}{\Delta x} \int_x^{x+\Delta x} f(s)ds$, By Mean value theorem, there exists $\xi \in [x, x+\Delta x]$ such that

$$\int_x^{x+\Delta x} f(s)ds = \Delta x f(\xi).$$

Therefore $\lim_{\Delta x \to 0} \frac{\phi(x+\Delta x) - \phi(x)}{\Delta x} = \lim_{\Delta x \to 0} f(\xi)$. Since f is continuous, $\lim_{\Delta x \to 0} f(\xi) = f(\lim_{\Delta x \to 0} \xi) = f(x)$. Thus $\phi'(x) = f(x)$. ///

Remark 1.0.1. *If f is integrable then ϕ is continuous.*

Now we ask the following important
Question: It is always not true that $\int_a^b f'(x)dx = f(b) - f(a)$.?
The answer is NO. For example, take $f(x) = x^2 \sin \frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$. Then f is differentiable on $[0,1]$. Here the derivatives at the end points are the left/right derivatives. It is easy to check that $f'(x) = 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2}$ for $x \in (0,1)$ and $f'(0) = 0$. Therefore f' is not bounded and so not integrable.

Second Fundamental Theorem

Definition 1.0.7. *A function $F(x)$ is called anti-derivative of $f(x)$, if $F'(x) = f(x)$.*

Theorem 1.0.8. *Suppose $F(x)$ is an anti- derivative of continuous function $f(x)$. Then $\int_a^b f(x)dx = F(b) - F(a)$.*

Proof. By First fundamental theorem, we have

$$\frac{d}{dx}\int_a^x f(s)ds = f(x).$$

Also $F'(x) = f(x)$. Hence $\int_a^x f(s)ds = F(x) + c$ for some constant $c \in \mathbb{R}$. Taking $x = a$, we get $c = -F(a)$. Now taking $x = b$ we get $\int_a^b f(x)dx = F(b) - F(a)$. ///

Moreover, one can prove the following more general theorem:

Theorem 1.0.9.

If f is integrable and if there exists F such that $F' = f$, then $\int_a^b f(x)dx = F(b) - F(a)$.

Proof. Proof follows from the neccessary and sufficient condition. Interested students may see the text book.

Problem 1.0.1. $\lim_{x\to 0} \frac{1}{x}\int_0^x \sin(t^2)dt = 0.$

Change of Variable formula

Theorem 1.0.10. *Let $u(t), u'(t)$ be continuous on $[a, b]$ and f is a continuous function on the interval $u([a, b])$. Then*

$$\int_a^b f(u(x))u'(x)dx = \int_{u(a)}^{u(b)} f(y)dy.$$

Proof. Note that $f([a, b])$ is a closed and bounded interval. Since f is continuous, it has primitive F. i.e., $F(x) = \int_a^x f(t)dt$. Then by chain rule of differentiation, $\frac{d}{dt}F(u(t)) = F'(u(t))u'(t)$. i.e., $F(u(t))$ is the primitive of $f(u(t))u'(t)$ and by Newton-Leibnitz formula, we get

$$\int_a^b f(u(t))u'(t)dt = F(u(b)) - F(u(a)).$$

On the other hand, for any two points in $u([a, b])$, we have (by Newton-Leibnitz formula)

$$\int_A^B f(y)dy = F(B) - F(A).$$

Hence $B = u(b)$ and $A = u(a)$.

Problem 1.0.2. *Evaluate $\int_0^1 x\sqrt{1 + x^2}dx$.*
Taking $u = 1 + x^2$, we get $u' = 2x$ and $u(0) = 1, u(1) = 2$. Then

$$\int_0^1 x\sqrt{1 + x^2}dx = \frac{1}{2}\int_1^2 \sqrt{u}du = \frac{1}{3}u^{\frac{2}{3}}\Big|_{u=1}^{2} = \frac{1}{3}(2^{\frac{2}{3}} - 1).$$