Lecture 34

2 Improper Integrals

In the previous section, we defined Riemann integral for functions defined on closed and bounded interval $[a, b]$. In this section our aim is to extend the concept of integration to the the following cases:

1. The function $f(x)$ defined on unbounded interval $[a, \infty)$ and $f \in \mathcal{R}[a, b]$ for all $b > a$.

2. The function is not defined at some points on the interval $[a, b]$.

We first consider
Improper integral of first kind: Suppose f is a bounded function defined on $[a, \infty)$ and $f \in \mathcal{R}[a, b]$ for all $b > a$.

Definition 2.0.1. *The improper integral of f on $[a, \infty)$ is defined as*

$$\int_a^\infty f(x)dx := \lim_{b \to \infty} \int_a^b f(x)dx.$$

If the limit exists and is finite, we say that the improper integral converges. If the limit goes to infinity or does not exist, then we say that the improper integral diverges.

Examples 2.0.2. *1. (i)* $\int_1^\infty \frac{1}{x^2}dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}dx = \lim_{b\to\infty} 1 - \frac{1}{b} = 1.$

2. (ii) $\int_0^\infty \frac{dx}{1+x^2} = \lim_{b\to\infty} \int_0^b \frac{dx}{1+x^2} = \lim_{b\to\infty} \arctan x\Big|_0^b = \frac{\pi}{2}.$

Theorem 2.0.3. ***Comparison test:*** *Suppose $0 \leq f(x) \leq \phi(x)$ for all $x \geq a$, then*

1. $\int_a^\infty f(x)dx$ *converges if* $\int_a^\infty \phi(x)dx$ *converges.*

2. $\int_a^\infty \phi(x)dx$ *diverges if* $\int_a^\infty f(x)dx$ *diverges.*

Proof. Define $F(x) = \int_a^x f(t)dt$ and $G(x) = \int_a^x g(t)dt$. Then by properties of Riemann integral, $0 \leq F(x) \leq G(x)$ and we are given that $\lim\limits_{x\to\infty} G(x)$ exists. So $G(x)$ is bounded. F is monotonically increasing and bounded above. Therefore, $\lim\limits_{x\to\infty} F(x)$ exists.

Examples 2.0.4. *1.* $\int_1^\infty \frac{dx}{x^2(1+e^x)}$. *Note that* $\frac{1}{x^2(1+e^x)} < \frac{1}{x^2}$ *and* $\int_1^\infty \frac{dx}{x^2}$ *converges.*

2. $\int_1^\infty \frac{x^3}{x+1}dx$. *Note that* $\frac{x^3}{x+1} > \frac{x^2}{2}$ *on* $[1, \infty)$ *and* $\int_1^\infty x^2 dx$ *diverges.*

Definition 2.0.5. *Let $f \in \mathcal{R}[a, b]$ for all $b > a$. Then we say $\int_a^\infty f(x)dx$ converges absolutely if $\int_a^\infty |f(x)|dx$ converges.*

In the following we show that absolutely convergence implies convergence of improper integral.

Theorem 2.0.6. *If the integral $\int_a^\infty |f(x)|dx$ converges, then the integral $\int_a^\infty f(x)dx$ converges.*

Proof. Note that $0 \le f(x) + |f(x)| \le 2|f(x)|$. So the improper integral $\int_a^\infty f(x) + |f(x)|dx$ converges by comparison theorem above. Also $\int_a^\infty |f(x)|$ converges. Therefore, $\int_a^\infty f(x)dx = \int_a^\infty f(x) + |f(x)|dx - \int_a^\infty |f(x)|dx$ also converges. ///

The converse of the above theorem is not true. For example take the integral $\int_\pi^\infty \frac{\sin x}{x}dx$. This integral does not converge absolutely. Indeed,

$$\begin{aligned}\int_\pi^\infty \frac{|\sin x|}{x}dx &= \sum_{n=1}^\infty \int_{n\pi}^{(n+1)\pi} \frac{|\sin x|}{x}dx \\ &\ge \sum_{n=1}^\infty \frac{1}{n\pi}\int_{n\pi}^{(n+1)\pi} |\sin x| \\ &= \sum_{n=0}^\infty \frac{1}{n\pi}\int_0^\pi \sin x = \frac{2}{\pi}\sum_{n=1}^\infty \frac{1}{n}.\end{aligned}$$

On the other hand, by integration by parts,

$$\begin{aligned}\lim_{b\to\infty}\int_1^b \frac{\sin x}{x}dx &= \lim_{b\to\infty}\int_1^b \frac{1}{x}d(1-\cos x) \\ &= \lim_{b\to\infty}\left(\frac{(1-\cos b}{b} + \int_1^b \frac{1-\cos x}{x^2}dx\right)\end{aligned}$$

It is not difficult to show that the limits on the right exist.

Examples 2.0.7. *1. $\int_1^\infty \frac{\sin x}{x^3}dx$. Easy to see that $|\frac{\sin x}{x^3}| \le |\frac{1}{x^3}|$ and $\int_1^\infty \frac{dx}{x^3}$ converges.*

2. $\int_0^\infty \frac{e^{-x^2}\sin x}{\log(1+x)}$. Here first note that $\lim_{x\to 0}\frac{e^{-x^2}\sin x}{\log(1+x)} = 1$. Therefore the integral is proper at $x = 0$. For $x > 10$(say):

$$|f(x)| \le \frac{e^{-x^2}}{\log(1+x)} < e^{-x^2} \le e^{-x}$$

Hence the integral $\int_{10}^\infty \frac{e^{-x^2}\sin x}{\log(1+x)}$ converges by comparison theorem.

Theorem 2.0.8. ***Limit comparison test:*** *Let $f(x), g(x)$ are defined and positive for all $x \ge a$ and $\lim_{x\to\infty}\frac{f(x)}{g(x)} = L$.*

1. *If $L \in (0, \infty)$, then the improper integrals $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ are either both convergent or both divergent. i.e., $\int_a^\infty f(x)dx$ converges $\iff$ $\int_a^\infty g(x)$ converges.*

2. *If $L = 0$, then $\int_a^\infty f(x)dx$ converges if $\int_a^\infty g(x)dx$ converges. i.e, $\int_a^\infty g(x)dx$ converges $\implies \int_a^\infty f(x)dx$ converges.*

3. *If $L = \infty$, then $\int_a^\infty f(x)dx$ diverges if $\int_a^\infty g(x)dx$ diverges. i.e., $\int_a^\infty g(x)dx$ diverges $\implies$ $\int_a^\infty f(x)dx$ diverges.*

Proof. From the definition of limits, for any $\epsilon > 0$, there exists $M > 0$ such that

$$x \geq M \implies L - \epsilon < \frac{f(x)}{g(x)} < L + \epsilon.$$

Thus for $x \geq M$, we have $(L - \epsilon)g(x) < f(x) < (L + \epsilon)g(x)$.
Now in case (1), since $L > 0$, we can find $\epsilon > 0$ such that $L - \epsilon > 0$. Using the property 3, it is enough to prove the convergence/divergence for x large, say $x \geq M$. In this interval, we have the comparison $(L - \epsilon)g(x) < f(x) < (L + \epsilon)g(x)$. Now integrating this, we get the result.
In case (2), we have $f(x) < (L + \epsilon)g(x)$. Again, integrate on both sides.
In case (3) by the definition, for every $M > 0$, there exists, R such that $f(x) > Mg(x)$ for all $x > R$. Now the result follows similar to (1) and (2).

Examples 2.0.9. 1. *$\int_1^\infty \frac{dx}{\sqrt{x+1}}$. Take $f(x) = \frac{1}{\sqrt{x+1}}$ and $g(x) = \frac{1}{\sqrt{x}}$. Then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$ and $\int_1^\infty g(x)dx$ diverges. So by above theorem, $\int_1^\infty f(x)dx$ diverges.*

2. *$\int_1^\infty \frac{dx}{1+x^2}$. Take $f(x) = \frac{1}{1+x^2}$ and $g(x) = \frac{1}{x^2}$. Then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$ and $\int_1^\infty g(x)dx$ converges. So by above theorem, $\int_1^\infty f(x)dx$ converges.*

3. *$\int_0^\infty \frac{x}{\cosh x}dx$. Let $f(x) = \frac{x}{\cosh x} = \frac{2xe^x}{e^{2x}+1} \sim xe^{-x}$. So choose $g(x) = xe^{-x}$. Then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 2$ and $\int_0^\infty g(x)$ converges.*

Improper integrals of second kind

Definition 2.0.10. *Let $f(x)$ be defined on $[a, c)$ and $f \in \mathcal{R}[a, c - \epsilon]$ for all $\epsilon > 0$. Then we define*

$$\int_a^c f(x)dx = \lim_{\epsilon\to 0} \int_a^{c-\epsilon} f(x)dx.$$

Then $\int_a^b f(x)dx$ is said to converge if the limit exists and is finite. Otherwise, we say improper integral $\int_a^b f(x)dx$ diverges.

Example: $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{\epsilon\to 0} \int_\epsilon^1 \frac{dx}{\sqrt{x}} = \lim_{\epsilon\to 0} 2(1 - \sqrt{\epsilon}) = 2.$

Suppose $a_1, a_2, ...a_n$ are finitely many discontinuities of $f(x)$ in $[a, b]$. Then

$$\int_a^b f(x)dx = \int_a^{a_1} f(x)dx + \int_{a_1}^{a_2} f(x)dx + \int_{a_2}^{a_3} f(x)dx + ... + \int_{a_n}^{b} f(x)dx$$

If all the improper integrals on the right hand side converge, then we say the improper integral of f over $[a, b]$ converges. Otherwise, we say it diverges.

The following comparison and Limit comparison tests can be proved following similar arguments:

Theorem 2.0.11. *(Comparison Theorem:) Suppose $0 \leq \phi(x) \leq f(x)$ for all $x \in [a, c)$ and are discontinuous at c.*

1. *If $\int_a^c f(x)dx$ converges then $\int_a^c \phi(x)dx$ converges.*
2. *If $\int_a^c \phi(x)dx$ diverges then $\int_a^c f(x)dx$ diverges.*

Problem 2.0.1. *Test the convergence of $\int_0^1 \frac{e^x}{\sqrt{x}}dx$*

Solution: $e^x < e$ for all $x \in (0, 1)$. Therefore $\frac{e^x}{\sqrt{x}} < \frac{e}{\sqrt{x}}$ and $\int_0^1 \frac{1}{\sqrt{x}}dx$ converges. Therefore $\int_0^1 \frac{e^x}{\sqrt{x}}dx$ also converges.

Theorem 2.0.12. *(Limit comparison theorem:) Suppose $0 < f(x), g(x)$ be continuous in $[a, c)$ and $\lim_{x\to c} \frac{f(x)}{g(x)} = L$. Then*

1. *If $L \in (0, \infty)$. Then $\int_a^c f(x)dx$ and $\int_a^c g(x)dx$ both converge or diverge together.*
2. *If $L = 0$ and $\int_a^c g(x)dx$ converges then $\int_a^c f(x)dx$ converges.*
3. *If $L = \infty$ and $\int_a^c g(x)dx$ diverges then $\int_a^c f(x)dx$ diverges.*

Proof. From the definition of limit, for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$x \in (c - \delta, c) \implies (L - \epsilon)g(x) < f(x) < (L + \epsilon)g(x)$$

Rest of the proof follows in the similar lines theorems on first kind, by choosing $\epsilon < L$.

Problem 2.0.2. *Test the convergence of the integral* $\displaystyle\int_0^1 \frac{e^{\sqrt{x}} - 1}{x}dx.$

Solution: Let $f(x) = \frac{e^{\sqrt{x}}-1}{x}$ and $g(x) = \sqrt{x}$. Then by Taylor's theorem (OR L'Hospital rule)

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{\sqrt{x}(e^{\sqrt{x}} - 1)}{x} = \lim_{x\to} \frac{\sqrt{x}(\sqrt{x} + o(x))}{x} = 1.$$

Also $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}dx$ converges. Therefore $f(x)$ is integrable.