Lecture 35

Transforming improper integrals:

Sometimes improper integrals may be transformed into proper integrals. For example consider the improper integral $I = \int_1^3 \frac{dx}{\sqrt{x}\sqrt{3-x}}$. Taking the transformation $y = \frac{1}{3-x}$, we get $I = \int_{1/2}^{\infty} \frac{dy}{y\sqrt{3y-1}}$. This is an improper integral of first kind. Instead, if we choose the transformation $3 - x = u^2$ then $I = \int_0^{\sqrt{2}} \frac{2udu}{u\sqrt{3-u^2}}$, which is a proper integral.

Remark 1.0.1. *It is important to note that the "symmetric" limit could be convergent but the limit may not exist. For example,*

$$\begin{aligned}\int_{-1}^{1} \frac{dx}{x^3} &= \int_{-1}^{0} \frac{dx}{x^3} + \int_0^1 \frac{dx}{x^3} \\ &= \lim_{\epsilon_1 \to 0} \int_{-1}^{-\epsilon_1} \frac{dx}{x^3} + \lim_{\epsilon_2 \to 0} \int_{\epsilon_2}^{1} \frac{dx}{x^3} \\ &= \frac{1}{2} \lim_{\epsilon_1, \epsilon_2 \to 0} (\frac{1}{\epsilon_1^2} - \frac{1}{\epsilon_2^2}), \end{aligned} \tag{1.1}$$

where $\epsilon_1, \epsilon_2 \to 0$ as $n \to \infty$. It is easy to see that if one takes $\epsilon_1 = \epsilon_2$, then the limit exits and is equal to 0. But if one takes $\epsilon_1 = \frac{1}{(n+1)^2}, \epsilon_2 = \frac{1}{n^2}$, then the above limit in (1.1) does not exist. So through different sequences, we are getting different limits. By now, by our familiarity with existence of limits, we say integral diverges.

Gamma and Beta functions:

Consider the *Gamma function* defined as improper integral for $p > 0$,

$$\Gamma(p) = \int_0^{\infty} x^{p-1} e^{-x} dx$$

This integral is improper of second kind in the neighbourhood of 0 as x^{p-1} goes to infinity as $x \to 0$ (when $p < 1$). Since the domain of integration is $(0, \infty)$, the integral is improper of first kind. To prove the convergence, we divide the integral into

$$\begin{aligned}\Gamma(p) =& \int_0^1 x^{p-1} e^{-x} dx + \int_1^{\infty} x^{p-1} e^{-x} dx \\ =& I_1 + I_2\end{aligned}$$

To see the convergence of I_1 we take $f(x) = x^{p-1}e^{-x}$ and $g(x) = x^{p-1}$, then $\lim_{x \to 0} \frac{f(x)}{g(x)} = 1$ and

$\int_0^1 x^{p-1}dx$ converges. To see the convergence of I_2, take $f(x) = x^{p-1}e^{-x}$ and $g(x) = \frac{1}{x^2}$. Then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} x^{2+p-1}e^{-x} = 0$ and $\int_1^\infty \frac{1}{x^2}dx$ converges. Hence by (2) of limit comparison theorem, the integral converges.

Next we consider the *Beta function* defined as improper integral for $p > 0, q > 0$,

$$\beta(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx$$

If $p > 1$ and $q > 1$, then the integral is definite integral. When $p < 1$ and/or $q < 1$, this integral is improper of second kind at 0 and/or 1. To prove the convergence, we divide as before

$$\begin{aligned}\int_0^1 x^{p-1}(1-x)^{q-1}dx &= \int_0^{1/2} x^{p-1}(1-x)^{q-1}dx + \int_{1/2}^1 x^{p-1}(1-x)^{q-1}dx \\ &= I_1 + I_2.\end{aligned}$$

To see the convergence of I_1, take $f(x) = x^{p-1}(1-x)^{q-1}$ and $g(x) = x^{p-1}$. Then $\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0}(1-x)^{q-1} = 1$ and $\int_0^{1/2} x^{p-1}dx$ converges. Similarly, for convergence of I_2, we take $f(x) = x^{p-1}(1-x)^{q-1}$ and $g(x) = (1-x)^{q-1}$.

Some identities of beta and gamma functions:

1. $\Gamma(1) = \int_0^\infty e^{-x}dx = 1$.

2. $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$.
   Integration by parts formula implies,

$$\Gamma(\alpha + 1) = \int_0^\infty x^\alpha e^{-x} = -(x^\alpha e^{-x})|_0^\infty + \alpha \int_0^\infty x^{\alpha-1}e^{-x}dx = \alpha\Gamma(\alpha).$$

   Therefore, $\Gamma(m + 1) = m!\ \forall m \in I\!N$.

3. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

$$\begin{aligned}\left(\Gamma(\frac{1}{2})\right)^2 =& 4\int_0^\infty \int_0^\infty e^{-u^2}e^{-v^2}dudv, \quad \text{take } u = r\cos\theta, v = r\sin\theta, \\ =& 4\int_0^\infty \int_0^{\pi/2} e^{-r^2}rdrd\theta = \pi\end{aligned}$$

4. $\beta(m,n) = \beta(n,m)$. Substituting $t = 1-x$ in the definition of $\beta(m,n)$, we get

$$\beta(m,n) = \int_0^1 t^{n-1}(1-t)^{m-1}dt = \beta(n,m)$$

5. $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta$
   Taking $x = \sin^2\theta$ in $\beta(m,n)$, we get

$$\beta(m,n) = \int_0^{\pi} \cos^{2m-2}\theta \sin^{2n-2}\theta \cos\theta \sin\theta d\theta = 2\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta d\theta.$$

6. $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

Problem 1.0.1. *Evaluate* (i) $\int_0^{\infty} x^{2/3}e^{-\sqrt{x}}dx$ $\quad (ii)$ $\int_0^1 x^{\frac{3}{2}}(1-\sqrt{x})^{\frac{1}{2}}dx$

For (i), take $t = \sqrt{x}$, then the given integral becomes

$$\int_0^{\infty} t^{4/3}e^{-t}2tdt = 2\int_0^{\infty} e^{-t}t^{7/3}dt = 2\Gamma(\frac{10}{3}) = \frac{56}{27}\Gamma(1/3).$$

For (ii), again take $t = \sqrt{x}$, then the integral becomes

$$2\int_0^1 t^3(1-t)^{1/2}tdt = 2\int t^4(1-t)^{1/2}dt = 2\beta(5,3/2) = 2\frac{\Gamma(5)\Gamma(3/2)}{\Gamma(13/2)} = \frac{512}{3465}$$

Cauchy Principal Value:

Consider the improper integral $I = \int_0^{\infty} \sin x dx$. It is easy to see from the definition that $I = \lim_{a\to\infty}(1-\cos a)$ which does not exist.Similarly, $\int_{-\infty}^0 \sin x dx$ does not exist. But

$$\lim_{c\to\infty}\int_{-c}^{c} \sin x dx$$

exists and is equal to 0. Though the improper integral does not exist, this symmetric limit exists. This is called *Cauchy Principal value* of improper integral

Definition 1.0.1. *The Cauchy Principal value of improper integral of first kind is defined as*

$$CPV\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)dx$$

For the improper integral of second kind, with $c \in (a,b)$ *as point of discontinuity of* $f(x)$ *as*

$$CPV\int_a^b f(x)dx = \lim_{\delta\to 0}\int_a^{c-\delta} f(x)dx + \int_{c+\delta}^{b} f(x)dx$$

Examples 1.0.2. *First kind:* $\int_{-\infty}^{\infty} x^{2n+1}dx$ *for all* $n = 1, 2, 3, \ldots$. *In this case it is easy to check that* $\lim_{a\to\infty} \int_{-a}^{a} x^{2n+1} = 0$. *But the the improper integrals* $\int_0^{\infty} x^{2n+1}$ *and* $\int_{-\infty}^{0} x^{2n+1}$ *does not converge.*

Second kind: $\int_{-1}^{1} x^{-(2n+1)}dx$, *for* $n = 1, 2, 3, \ldots$. *Simply evaluate*

$$\int_{-1}^{-\epsilon} x^{-(2n+1)} + \int_{\epsilon}^{1} x^{-(2n+1)}$$

to see that the the limit is 0.

Integrals dependent on a Parameter

Consider an integral

$$I(\alpha) = \int_a^b f(x, \alpha)dx$$

where the integrand is depend on the parameter α. At times we can differentiate under the integral sign to evaluate the integral. It is sometimes not possible and leads to wrong assertions. For example, we know that $I = \int_0^{\infty} \frac{\sin x}{x} = \frac{\pi}{2}$. It is easy to notice with change of variable formula, taking $tx = y$, that $I = I(t) = \int_0^{\infty} \frac{\sin(tx)}{x} = \frac{\pi}{2}$. Now differentiating this, taking derivative inside integral, we get $I'(t) = \int_0^{\infty} \cos(tx)dx = 0$, which doesn't make sense.
Here we have a theorem, which explains under which conditions we can do the differentiation under integral sign.

Theorem 1.0.3. *Suppose,*

1. *Suppose* $f, \frac{d}{d\alpha}f(x, \alpha)$ *are continuous functions for* $x \in [a, b]$ *and* α *in an interval of containing* α_0.

2. $|f(x, \alpha)| \leq A(x), |\frac{d}{d\alpha}f(x, \alpha)| \leq B(x)$ *such that* a, b *are integrable on* $[a, b]$. *If the domain in unbounded, then the improper integrals* $\int_a^b Adx, \int_a^b Bdx$ *converge.*

Then I *is differentiable, and*

$$I'(\alpha) = \int_a^b \frac{d}{d\alpha}f(x, \alpha)dx.$$

Proof. .

$$\begin{aligned}\frac{d}{d\alpha}I(\alpha) &= \lim_{\Delta\alpha\to 0} \frac{I(\alpha + \Delta\alpha) - I(\alpha)}{\Delta\alpha} \\ &= \lim \frac{1}{\Delta\alpha}\left[\int_a^b (f(x, \alpha + \Delta\alpha) - f(x, \alpha))dx\right]\end{aligned}$$

Now by Taylor's theorem, $f(x,\alpha+\Delta\alpha)-f(x,\alpha)=\Delta\alpha\frac{d}{d\alpha}f(x,\alpha+\theta\Delta\alpha)$. Since $\frac{d}{d\alpha}f(x,\alpha)$ is continuous, we have $\frac{d}{d\alpha}f(x,\alpha+\theta\Delta\alpha)=\frac{d}{d\alpha}f(x,\alpha)+\epsilon$, where $0<\theta<1$ and $\epsilon\to 0$ as $\Delta\alpha\to 0$. Thus

$$\frac{d}{d\alpha}I(\alpha)=\lim_{\Delta\alpha\to 0}\int_a^b\frac{d}{d\alpha}f(x,\alpha)+\epsilon=\int_a^b\frac{d}{d\alpha}f(x,\alpha)dx$$

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In fact the following holds.

Newton-Leibnitz Formula:

Let $h(x)=\displaystyle\int_{a(x)}^{b(x)}f(x,t)dt$. Then $h'(x)=\displaystyle\int_{a(x)}^{b(x)}\frac{df}{dx}(x,t)dt+f(x,b(x))b'(x)-f(x,a(x))a'(x)$

Examples 1.0.4. 1. *Evaluate* $I(\alpha)=\displaystyle\int_0^\infty e^{-x}\frac{\sin\alpha x}{x}dx$.

By the above formula, $I'(\alpha)=\int_0^\infty e^{-x}\cos\alpha x dx=\frac{1}{1+\alpha^2}$. *Therefore,* $I(\alpha)=\arctan\alpha+C$. *Also* $I(0)=\int_0^\infty e^{-x}\sin 0x=0$. *Hence* $C=0$.

2. *Test the convergence and evaluate the integral* $\displaystyle\int_0^\infty e^{\frac{1}{2}(t^2-x^2)}\cos(tx)dx$.

$$|I|\le e^{t^2/2}\int_0^\infty|e^{-x^2}\cos(tx)|dx\le C\int_0^\infty e^{-x^2}dx=\frac{1}{2}\Gamma(\frac{1}{2}).$$

Hence the integral converges. By Newton Leibnitz formula,

$$\begin{aligned}I_a'(t)&=\int_0^a e^{\frac{1}{2}(t^2-x^2)}(t\cos tx-x\sin tx)dx\\&=\int_0^a\frac{\partial}{\partial x}e^{\frac{1}{2}(t^2-x^2)}\sin tx dx\\&=e^{\frac{1}{2}(t^2-a^2)}\sin at\end{aligned}$$

Therefore, $I'(t)=\lim_{a\to\infty}I_a'(t)=0$. *Now note that* $I(0)=\int_0^\infty e^{-x^2/2}dx=\frac{1}{\sqrt{2}}\Gamma(1/2)$. *Hence* $I(t)=\sqrt{\frac{\pi}{2}}$.