Lecture 36

1 Multiple Integrals

1.1 Double integrals

Let $f(x, y)$ be a real valued function defined over a domain $\Omega \subset I\!R^2$. To start with, let us assume that Ω be the rectangle $R = (a, b) \times (c, d)$. We partition the rectangle with node points (x_k, y_k), where

$$a = x_1 < x_2 < ...., x_n = b, \text{ and } c = y_1 < y_2 < ...., y_n = d.$$

Let R_{nm} be the small sub-rectangle with above vertices. Now we can define Upper and lower Riemann sum as

$$U(P_n, f) = \sum_{n,m} \sup_{R_{nm}} f(x, y)|R_{nm}|$$

and

$$L(P_n, f) = \sum_{n,m} \inf_{R_{nm}} f(x, y)|R_{nm}|$$

where $|R_{nm}|$ is the area of the rectangle R_{nm}. Here we may define the norm of partition P_n as $\|P_n\| = \max_i \sqrt{|x_i - x_{i-1}|^2 + |y_i - y_{i-1}|^2}$. Then by our understanding of definite integral we can define the upper, lower integrals and $f(x, y)$ is integrable if and only if

$$\inf\{U(P, f) : P\} = \sup\{L(P, f) : P\}.$$

The definite integral is defined as

$$\iint_\Omega f(x, y)dA = \inf\{U(P, f) : P\} = \sup\{L(P, f) : P\}.$$

It can be shown that $\iint_\Omega f(x, y)dA = \lim_{\|P_n\| \to 0} S(P_n, f)$, where $S(P_n, f) = \sum_{n,m} f(x_k, y_k)|R_{nm}|$.

We have the following **Fubini's theorem** for rectangle:
Suppose $f(x, y)$ is integrable over $R = (a, b) \times (c, d)$, then

$$\iint_R f(x, y)dA = \int_a^b \left(\int_c^d f(x, y)dy \right) dx = \int_c^d \left(\int_a^b f(x, y)dx \right) dy$$

In case of $f(x, y) \geq 0$ we may interpret this as the volume of the solid formed by the surface $z = f(x, y)$ over the rectangle R. This is precisely the "sum" of areas of the cross section $A(x) = \int_c^d f(x, y)dy$ between $x = a$ and $x = b$. Since x varies over all of (a, b), this sum is nothing but the integral $\int_a^b A(x)dx$.

For any general bounded domain Ω, we can divide the domain into small sub domains Ω_k and consider the upper, lower sum exactly as above by replacing R_{nm} by Ω_k. Then a function $f(x, y) : \Omega \to I\!R$ is integrable if the supremum of lower sums and infimum of upper sums exist and are equal. We may define

$$\iint_\Omega f(x, y)dA = \lim_{\|P_n\|\to 0} \sum_k f(x_k, y_k)|\Omega_k|$$

The **basic properties** of the definite integral like integrability of $f \pm g$, kf and domain decomposition theorems holds in this case also.

We call a domain as y**-regular** if $\Omega = \{(x, y) : a \le x \le b, g_1(x) \le y \le g_2(x)\}$ for some continuous functions g_1, g_2. Similarly, Ω is x**-regular** if $\Omega = \{(x, y) : c \le y \le d, h_1(y) \le x \le h_2(y)\}$. A domain is called regular if it is either x-regular or y-regular. Then we have the following Fubini's theorem for regular domains.

Theorem 1.1.1 *Let $f(x, y)$ be continuous over Ω.*

1. *If Ω is y-regular, i.e.,$\Omega = \{(x, y) : a \le x \le b, g_1(x) \le y \le g_2(x)\}$. Then*

$$\iint_R f(x, y)dA = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x, y)dy \right) dx.$$

2. *If Ω is x-regular, i.e.,$\Omega = \{(x, y) : c \le y \le d, h_1(y) \le x \le h_2(y)\}$. Then*

$$\iint_R f(x, y)dA = \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x, y)dx \right) dy.$$

This theorem basically says that if a function is integrable over a domain Ω, then the value of integral is does not depend on order of integration. That is we can integrate with respect to x first followed by y or vice versa.

Example 1.1.1 *Evaluate the integral $\iint_\Omega (x + y + xy)dA$ where Ω is the triangle bounded by $y = 0, x = 1$ and $y = x$.*

Solution: The triangle is regular in both x and y. The given triangle is

$$\{(x, y) : 0 \le x \le 1, 0 \le y \le x\} = \{(x, y) : 0 \le y \le x, y \le x \le 1\}$$

Therefore, taking it as $y-$ regular we see that the domain is bounded below by $y = g_1(x) = 0$ and above by $y = g_2(x) = x$ over $x = 0$ to $x = 1$. Hence,

$$\begin{aligned}\iint_\Omega (x + y + xy)dA &= \int_{x=0}^{x=1} \left(\int_{y=0}^{x} (x + y + xy)dy \right) dx \\ &= \int_0^1 (x^2 + \frac{x^2}{2} + \frac{x^3}{2})dx = \frac{15}{24}\end{aligned}$$

Similarly, taking it as $x-$ regular, we see that the domain is bounded below y $x = h_1(y) = y$ and above by $x = h_2(y) = 1$ over $y = 0$ to 1. Hence

$$\iint_\Omega (x+y+xy)dA = \int_{y=0}^1 \left(\int_{x=y}^1 (x+y+xy)dx \right) dy = \frac{15}{24}$$

Example 1.1.2 *Evaluate the integral* $\iint_\Omega (2+4x)dA$ *where* Ω *is the domain bounded by* $y = x$ *and* $y = x^2$.

Solution:

$$\begin{aligned}\iint_\Omega (2+4x)dA &= \int_{x=0}^1 \left(\int_{y=x^2}^x (2+4x)dy \right) dx \\ &= \int_0^1 (2x + 2x^2 - 4x^3)dx = 2/3\end{aligned}$$

On the other hand, this is also equal to $\displaystyle\int_{y=0}^1 \left(\int_{x=y}^{\sqrt{y}} (2+4x)dx \right) dx.$

Remark 1.1.1

1. *When*$f(x,y) = 1$, *then we approximate the area of* Ω *as* $A \sim \sum_k \Omega_k = \sum_k f(x_k, y_k)|\Omega_k|$ *where* $f = 1$. *By the definition of Riemann integral this sum converges to* $\iint_\Omega dA$ *as* $\|P_n\| \to 0$.
2. *As discussed in the beginning, when* $f(x,y) \geq 0$, *the* $\iint_\Omega f(x,y)dA$ *is the volume of the solid bounded above by* $z = f(x,y)$ *and below by* Ω.

Problem 1.1.1 *Find the area bounded by* $y = 2x^2$ *and* $y^2 = 4x$.

Solution: The two parabola's intersect at $(0,0)$ and $(1,1)$. Hence the area is

$$A = \int_0^1 \int_{2x^2}^{2\sqrt{x}} dydx = \int_0^1 (2\sqrt{x} - 2x^2)dx = \frac{2}{3}.$$

Problem 1.1.2 *Find the volume of the solid under the paraboloid* $z = x^2 + y^2$ *over the bounded domain* R *bounded by* $y = x, x = 0$ *and* $x + y = 2$.

Solution: The domain of integration R is Y-regular bounded above by $x + y = 2$ and below by $y = x$ with x varying over $(0,1)$.

$$\begin{aligned}V = \iint_R (x^2+y^2)dA &= \int_0^1 \left(\int_{y=x}^{y=2-x} (x^2+y^2)dy \right) dx \\ &= \int_0^1 \frac{y^3}{3} + yx^2 \Big|_{y=x}^{y=2-x} dx = \int_0^1 (\frac{1}{3}((2-x)^3 - x^3) + x^2(2-2x))dx\end{aligned}$$

Problem 1.1.3 *Find the volume of the solid bounded above by the surface* $z = x^2$ *and below by the plane region* R *bounded by the parabola* $y = 2 - x^2, y = x$.

Solution: The points of intersection of $y = x, y = 2 - x^2$ are $x = -2, 1$. So $R = \{(x, y) : -2 \leq x \leq 1, x \leq y \leq 2 - x^2\}$. Therefore,

$$V = \int_{x=-2}^{1} \int_{y=x}^{2-x^2} x^2 dydx = \int_{-2}^{1} x^2(2 - x^2 - x)dx$$

Change of order

Consider the evaluation of integral $\iint_R \frac{\sin x}{x} dA$ over the triangle formed by $y = 0, x = 1$ and $y = x$. Since the can be extended as continuous function over R, by the basic properties of Riemann integral the function is integrable. Now by Fubini's theorem, the value of integral does not depend on the order of integration. As we noted earlier R is regular in x and y. If we take it as x regular, then $R = \{(x, y) : 0 \leq y \leq 1, y \leq x \leq 1\}$ and try to evaluate the integral, then

$$\iint_R \frac{\sin x}{x} dA = \int_0^1 \left(\int_{x=y}^{1} \frac{\sin x}{x} dx \right) dy.$$

This is singular integral and difficult to evaluate.
But when we consider R to be y-regular, we see that $R = \{(x, y) : \ 0 \leq x \leq 1, 0 \leq y \leq x\}$. Then the given integral is

$$\iint_R \frac{\sin x}{x} dA = \int_0^1 \int_{y=0}^{x} \frac{\sin x}{x} dydx = \int_0^1 \sin x dx = 1 - \cos 1$$

At times this technique can be used to evaluate some complicated definite integrals, for example,

Problem 1.1.4 *Evaluate the integral* $\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx, \ a, b > 0.$

Solution: This integral is equivalent to

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx = \int_0^\infty \left(\int_a^b e^{-xy} dy \right) dx$$

The domain of integration is the infinite strip $\{(x, y) : 0 \leq x \leq \infty, \ a \leq y \leq b\}$. Changing the order of integration, we get

$$\begin{aligned} \int_0^\infty \left(\int_a^b e^{-xy} dy \right) dx &= \int_{y=a}^{b} \left(\int_0^\infty e^{-xy} dx \right) dy \\ &= \ln \frac{b}{a} \end{aligned}$$

1.2 Double integrals in Polar form

Suppose we are given a bounded region whose boundaries are given by polar equations, say $f_1(r, \theta) = 0, f_2(r, \theta) = 0$. Then we divide the region into smaller "polar rectangles" whose sides have constant r, θ values.

Suppose $f(r,\theta)$ is defined over a region R defined using the polar equations, $R : \alpha \leq \theta \leq \beta, g_1(\theta) \leq r \leq g_2(\theta)$. Then we divide the r range by $\Delta r, 2\Delta r, ..., m\Delta r$ and $\alpha, \alpha+\Delta\theta, ...\alpha + m'\Delta\theta = \beta$. Let ΔA be the polar rectangle with sides $r_k - \Delta r/2, r_k + \Delta r/2$ and $\alpha + k\Delta\theta, \alpha + (k+1)\Delta\theta$. Then we define the Riemann sum as

$$S_n = \sum_k f(r_k, \theta_k)\Delta A_k.$$

The area of small "polar rectangle" A_k is

$$\Delta A_k = \text{ area of outer sector} - \text{area of inner sector} = r_k \Delta r \Delta\theta.$$

As $\|P_n\| \to 0$,we get

$$S_n = \sum_k f(r_k, \theta) r_k \Delta r \Delta\theta \to \iint_R f(r,\theta) r dr d\theta.$$

Problem 1.2.1 *Find the area common to the cardioids* $r = 1 + \cos\theta$ *and* $r = 1 - \cos\theta$.

Solution: Since the region is symmetric with respect to x -axis and y-axis, the required area is

$$\begin{aligned} A =& 4\int_{\theta=0}^{\pi/2} \int_{r=0}^{1-\cos\theta} r dr d\theta \\ =& 4\int_0^{\pi/2} \frac{1}{2}(1 - 2\cos\theta + \cos^2\theta) d\theta = 2(\frac{\pi}{2} - 2 + \int_0^{\pi/2} \cos^2\theta d\theta) \end{aligned}$$

Problem 1.2.2 *Evaluate* $\iint_R 3y dA$ *where* R *is the region bounded below by* x*-axis and above by the cardioid* $r = 1 - \cos\theta$.

Solution: The given integral is equivalent to

$$\iint_R 3y dA = \int_{\theta=0}^{\pi} \int_{r=0}^{1-\cos\theta} r \sin\theta r dr d\theta.$$

Problem 1.2.3 *Evaluate* $I = \int_0^\infty e^{-x^2} dx$.

Solution: Recall that $2I = \Gamma(\frac{1}{2})$. Using the Fubini's theorem, we may write

$$I^2 = \left(\int_0^\infty e^{-x^2} dx\right)\left(\int_0^\infty e^{-y^2} dy\right) = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dA$$

Where the integration is over the first quadrant $(0,\infty) \times (0,\infty)$. So representing this in polar form we integrate over $\{(r,\theta) : 0 \leq r < \infty, 0 \leq \theta \leq \frac{\pi}{2}\}$. Therefore, the above integral becomes

$$\int_0^{\pi/2} \int_0^\infty e^{-r^2} r dr d\theta = \frac{\pi}{4}.$$