Lecture 37

1 Voume integrals and change of varibles

1.1 Triple (Volume) integrals

Let $f(x, y, z)$ be a real valued function defined over a closed and bounded region of space $\mathbb{R}^3$. For example the solid ball or rectangular box. Now we want to define the definite integral of $f(x, y, z)$ over such regions.
We partition the region by small planes parallel to the coordinate axes. Then we obtain small rectangular cubes over which the function will be approximated by $f(x_k y_k, z_k)$. We form the Riemann sum

$$S_n = \sum_k f(x_k, y_k, z_k)|\Omega_k|,$$

where $|\Omega_k|$ is the volume of the small rectangle. Now by our understanding of Riemann sums we choose refinement of partitions in such way that $\max_k |\Omega_k| \to 0$. Then we obtain the definite integral as

$$\iiint_\Omega f(x, y, z)dV = \lim_{n\to\infty} S_n.$$

Evaluation of integrals in three dimensions is done again using Fubini's theorem. In this case again Fubini's theorem states

Theorem 1.1.1 *Suppose $f(x, y, z)$ is integrable over $\Omega \subset \mathbb{R}^3$, then*

$$\begin{aligned}\iiint_\Omega f(x,y,z)dV &= \int_x\int_y\int_z f(x,y,z)\ dz\ dy\ dx = \int_x\int_z\int_y f(x,y,z)\ dx\ dz\ dy\\ &\int_z\int_x\int_y f(x,y,z)\ dy\ dx\ dz = \int_z\int_y\int_x f(x,y,z)\ dx\ dy\ dz\\ &= \int_y\int_x\int_z f(x,y,z)\ dz\ dx\ dy = \int_y\int_z\int_x f(x,y,z)\ dy\ dz\ dx\end{aligned}$$

To evaluate the triple integrals we follow the following steps:

1. Draw a line parallel to z axis that passes through the point (x, y) of R where R is the projection of Ω onto $\mathbb{R}^2$.

2. Identify the upper surface and lower surface through which this line passes at most once.

3. Identify the upper curve and lower curve of the projection R and limits of integration.

It is easy to see from the definition, the volume of Ω is

$$V = \lim_{k\to\infty}\sum_k |\Omega_k| = \lim_{k\to\infty}\sum_k 1\ |\Omega_k| = \iiint_\Omega 1\ dV$$

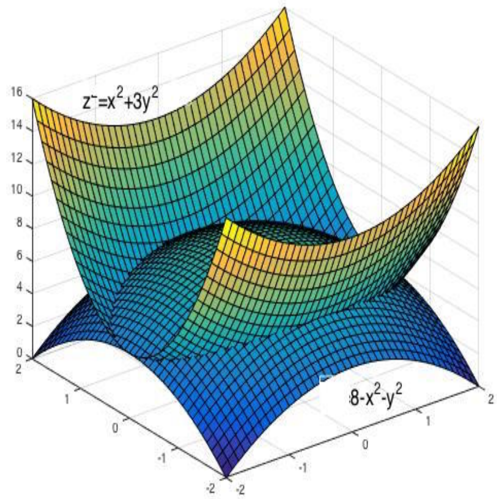

Figure 1: Volume bounded by the surfaces

Problem 1.1.1 *Find the volume of the region bounded by the surfaces* $z = x^2 + 3y^2$ *and* $z = 8 - x^2 - y^2$.

Solution: The volume is $V = \iiint_\Omega dzdydx$, where Ω is bounded above by the surface $z = 8 - x^2 - y^2$ and below by the surface $z = x^2 + 3y^2$. Therefore, the limits of z are from $z = x^2 + 3y^2$ to $z = 8 - x^2 - y^2$.

The Projection of Ω on xy-plane is the solution of

$$8 - x^2 - y^2 = x^2 + 3y^2 \quad \Longrightarrow \; x^2 + 2y^2 = 4.$$

Therefore the limits of x and y are to be determined by $R : x^2 + 2y^2 = 4$. Hence

$$\begin{aligned} V &= \iint_R \int_{y=x^2+3y^2}^{8-x^2-y^2} dzdA \\ &= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} (8 - 2x^2 - 4y^2) dydx \\ &= \int_{-2}^{2} \left((8 - x^2)y - \frac{4}{3} y^3 \right)_{y=-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \\ &= \frac{4\sqrt{2}}{3} \int_{-2}^{2} (4 - x^2)^{3/2} dx = 8\pi\sqrt{2}. \end{aligned}$$

Problem 1.1.2 *Find the volume of the region bounded by* $x + z = 1$, $y + 2z = 2$ *in the first quadrant.*

Solution: Draw line parallel to z-axis and note that the upper surfaces are: $2z + y = 2$ over triangle bounded by $x = 0, y = 1y = 2x$ and $z = 1 - x$ over the triangle bounded by $y = 0, x = 1, y = 2x$. Therefore,

$$V = \int_{y=0}^{2} \int_{x=0}^{y/2} \int_{z=0}^{\frac{2-y}{2}} dz\ dx\ dy + \int_{x=0}^{1} \int_{y=0}^{2x} \int_{z=0}^{1-x} dz\ dy\ dx$$

On the other hand, by first drawing the line parallel to x-axis, we get

$$V = \int_{z=0}^{1} \int_{y=0}^{2-2z} \int_{x=0}^{1-z} dx\ dy\ dz$$

Taking the line parallel to y-axis we get

$$V = \int_{x=0}^{1} \int_{z=0}^{1-x} \int_{y=0}^{2-2z} dy\ dz\ dx$$

Example 1.1.1 *(Order of integration) Evaluate* $\int_{z=0}^{4} \int_{y=0}^{1} \int_{x=2y}^{2} \frac{2\cos(x^2)}{\sqrt{z}} dx\ dy\ dz.$

Note that the projection of Ω onto xyplane is the triangle bounded by $y = 0, x = 2$ and $x = 2y$. So changing the order of integration in x and y, we get

$$I = \int_{z=0}^{4} \int_{x=0}^{2} \int_{y=0}^{x/2} \frac{2\cos(x^2)}{\sqrt{z}} dy\ dx\ dz.$$
$$= \int_{z=0}^{4} \int_{x=0}^{2} \frac{x\cos(x^2)}{\sqrt{z}} dx\ dz = 2\sin 4.$$

1.2 Substitutions in multiple integrals

Suppose a domain G in uv-plane is transformed onto a domain Ω of xy-plane by a transformation $x = g(u,v), y = h(u,v)$. Then any function of x, y may be written as a function of u, v. Then the relation between the double integral over G and Ω is

$$\iint_{\Omega} f(x,y) dx\ dy = \iint_{G} f(g(u,v), h(u,v)) |J(u,v)| du\ dv$$

where J is the Jacobian given by

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

The main idea of the proof is as follows. Let $(u,v), (u+\Delta u, v), (u+\Delta u, v+\Delta v)$ and $(u, v+\Delta v)$ be the vertices of the rectangle in the $uv-$plane. Let ΔA_k be its area element. Under the transformation this points are mapped to $(x_1, y_1) = (g(u,v), h(u,v)), (x_2, y_2) = (g(u+\Delta u, v), h(u+\Delta u, v)), (x_3, y_3) = (g(u+\Delta u, v+\Delta v)$ and $(x_4, y_4) = (g(u, v+\Delta v), h(u, v+\Delta v))$. Then by Taylor's theorem

$$g(u+\Delta u, v) = g(u,v) + \frac{\partial g}{\partial u} \Delta u + o((\Delta u)^2)$$

$$g(u+\Delta u, v+\Delta v) = g(u,v) + \frac{\partial g}{\partial u}\Delta u + \frac{\partial g}{\partial v}\Delta v + o((\Delta u)^2) + o((\Delta v)^2)$$

Then the area of the "rectangle" in xy-plane $\Delta\tilde{A_k}$ is

$$\begin{aligned}\Delta\tilde{A_k} \approx & |(x_3-x_1)(y_3-y_1)-(x_3-x_2)(y_3-y_2)| \\ \approx & |J|\Delta u\Delta v + o((\Delta u)^2) + o((\Delta v)^2)\end{aligned}$$

Taking this as the area in the Riemann sum of $f(x,y)$ we get the required formula.

Example 1.2.1 *Evaluate the integral* $I = \int_0^4 \int_{y/2}^{1+\frac{y}{2}} \frac{2x-y}{2} dxdy.$

Solution: The domain of integration is a parallelogram with vertices $(0,0),(1,0),(3,4)$ and $(2,4)$. One has to divide the domain into 3 domains. Instead we can take the transformation $u = \frac{2x-y}{2}, v = \frac{y}{2}$. Then the inverse transformation is $x = u+v, y = 2v$. Then

$$J = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2.$$

Under this transformation, the parallelogram is transformed into cube with vertices $(0,0),(1,0)(1,2)$ and $(0,2)$. Now by change of variable formula

$$I = \iint f(u+v, 2v)2dudv = \int_0^2 \int_0^1 2udu\ dv = 2.$$

Example 1.2.2 *Evaluate the integral* $I = \int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2 dA.$

Solution: The given domain is the triangle bounded by $x=0, y=0$ and $x+y=1$. In this case the integrand is complicated....so we can take transformation $u = x+y$ and $v = y-2x$. Under this transformation, the given triangle will be transformed into triangle bounded by $v = u, v = -2u$ and $u = 1$. The inverse of this transformation is $x = \frac{u-v}{3}$ and $y = \frac{2u+v}{3}$. Hence the Jacobian

$$J = \begin{vmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{vmatrix} = 1/3.$$

Hence

$$I = \int_0^1 \int_{v=-2u}^{u} \sqrt{u}v^2 dv\ du$$

Example 1.2.3 *Evaluate the integral* $I = \iint_R \frac{dA}{(2-x^2-y^2)^2}$ *over* $R: x^2+y^2 \leq 1.$

Solution: Taking the transformation $x = r\cos\theta, y = r\sin\theta$, we get

$$J = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

By substitution formula,

$$I = \int_0^{2\pi} \int_{r=0}^{1} \frac{r\ dr\ d\theta}{(2-r^2)^2} = 2\pi \int_1^2 \frac{dt}{2t^2} = \pi/2.$$

Substitution formula for triple integrals

As discussed above suppose a three dimensional domain G is transformed onto a domain D with a transformation $x = x(u,v,w), y = y(u,v,w), z = z(u,v,w)$, then

$$\iiint_D f(x,y,z)dV = \iiint_G F(u,v,w)|J(u,v,w)|dV$$

where

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}.$$

Example 1.2.4 *Evaluate* $\iiint_\Omega (x^2y + 3xyz)dV$ *where* $R = \{(x,y,z) : 1 \le x \le 2, 0 \le xy \le 2, 0 \le z \le 1\}$.

solution: We take the transformation $u = x, v = xy$ and $w = z$. Then the planes $x = 1, 2$ transforms to $u = 1, 2$. The plane $y = 0$ transforms to $v = 0$. The surface $xy = 2$ transforms to $v = 2$. Then the Jacobian J is

$$\frac{1}{J} = \begin{vmatrix} 1 & 0 & 0 \\ y & x & 0 \\ 0 & 0 & 1 \end{vmatrix} = x = u.$$

Now by substitution formula,

$$\begin{aligned} I &= \int_{u=1}^{2} \int_{v=0}^{2} \int_{w=0}^{1} (uv + 3vw)\frac{1}{u} dw\ dv\ du \\ &= \int_1^2 \int_0^2 (v + \frac{3v}{2u}) dv\ du \\ &= \int_1^2 (2 + \frac{3}{u}) du = 2 + 3\ln 2. \end{aligned}$$