Lecture 38

1 Cylindrical and Spherical coordinates

1.1 Cylindrical coordinates

A point P in the space ($\mathbb{R}^3$) is represented by (r, θ, z) where r, θ are polar coordinates of the projection of P on to xy-plane and z is the z distance of the projection from P. When we take the transformation $x = r\cos\theta, y = r\sin\theta, z = z$, the Jacobian is

$$J = \begin{vmatrix} x_r & x_\theta & x_z \\ y_r & y_\theta & y_z \\ z_r & z_\theta & z_z \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

Example 1.1.1 *Find the volume of the cylinder* $x^2 + (y-1)^2 = 1$ *bounded by* $z = x^2 + y^2$ *and* $z = 0$.

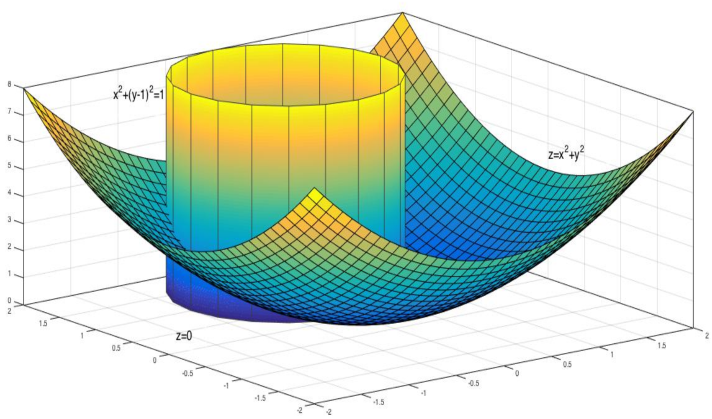

Figure 1: Volume bounded.

Solution: Drawing a line parallel to z axis, we see that the limits of z are from 0 to $x^2 + y^2$ and the projection onto xy-plane is the disc:$R : x^2 + (y-1)^2 \leq 1$. Therefore,

$$V = \iint_R \int_{z=0}^{x^2+y^2} dz dA$$

Now taking the cylindrical coordinates $x = r\cos\theta, y = r\sin\theta, z = z$ we get the projection to be

$$r^2\cos^2\theta + r^2\sin^2\theta - 2r\sin\theta = 0$$

$$i.e., \quad r(r - 2\sin\theta) = 0 \implies r = 0 \text{ to } r = 2\sin\theta$$

$$
\begin{aligned}
V &= \int_{\theta=0}^{\pi}\int_{r=0}^{2\sin\theta}\int_{z=0}^{r^2} r dz\ dr\ d\theta \\
&= \int_{\theta=0}^{\pi}\int_{r=0}^{2\sin\theta} r^3 dr\ d\theta \\
&= 4\int_{0}^{\pi} \sin^4\theta d\theta = \frac{5\pi}{4}
\end{aligned}
$$

1.2 Spherical polar coordinates

A point P in the space is represented by (ρ, θ, ϕ) where ρ is the distance of P from the origin, ϕ is the angle made by the ray OP with positive z axis and θ is the angle made by the projection of P (onto xy-plane) with positive x-axis. So it is not difficult to see that the relation with cartesian coordinates: The projection of P on xy-plane has polar representation: $x = r\cos\theta, y = r\sin\theta$ where r is the distance of the projected point to origin. Therefore $r = \rho\sin\phi$. From the definition of ϕ it is easy to see that $z = \rho\cos\phi$ and

$$
x = \rho\sin\phi\cos\theta,\ y = \rho\sin\phi\sin\theta,\ z = \rho\cos\phi
$$

The Jacobian in this case is

$$
J = \begin{vmatrix} x_\rho & x_\theta & x_\phi \\ y_\rho & y_\theta & y_\phi \\ z_\rho & z_\theta & z_\phi \end{vmatrix} = \begin{vmatrix} \rho\sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \rho\sin\phi\sin\theta & r\cos\theta\sin\phi & 0 \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix} = \rho^2\sin\phi
$$

To find limits of integration in spherical coordinates,

1. Draw a ray from the origin to find the surfaces $\rho = g(\theta, \phi), \rho = g_2(\theta, \phi)$ where it enters the region and leaves the region.

2. Rotate this ray away and towards z-axis to find the limits of ϕ

3. Identify the projection R of the domain on the xy-plane and polar form of R to write the limits of θ.

Example 1.2.1 *Evaluate* $\displaystyle\iiint_\Omega \frac{dV}{\sqrt{1+x^2+y^2+z^2}}$ *where* Ω *is the unit ball* $x^2+y^2+z^2 \le 1$.

Solution: Going to spherical polar coordinates $x = \rho\sin\phi\cos\theta,\ y = \rho\sin\phi\sin\theta,\ z = \rho\cos\phi$, we get

$$
\begin{aligned}
I &= \int_{\rho=0}^{1}\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} \frac{\rho^2}{\sqrt{1+\rho^2}} \sin\phi\ d\phi\ d\theta\ d\rho \\
&= (2\pi\times 2)\int_0^1 \frac{\rho^2}{\sqrt{1+\rho^2}} d\rho = 4\pi(\sqrt{2} - \frac{1}{2}\ln(\sqrt{2}+1)).
\end{aligned}
$$

Example 1.2.2 *Evaluate* $I = \displaystyle\iiint_\Omega x dV$ *where* Ω *is the part of the ball* $x^2+y^2+z^2 \le 4$ *in the first octant.*

Solution: Going to cylindrical coordinates, $x = \rho \sin\phi \cos\theta,\ y = \rho \sin\phi \sin\theta,\ z = \rho\cos\phi$. Since the domain is the ball of radius 4, we see that the limits of ρ are from 0 to 2. Again since it is cut by the xy-plane below, ϕ varies from 0 to $\pi/2$. The projection is the circle in the first quadrant with radius 2. So θ varies from 0 to $\pi/2$. Hence,

$$
\begin{aligned}
I =& \int_{\rho=0}^{2} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/2} \rho \sin\phi \cos\theta \ (\rho^2 \sin\phi) \ d\rho \ d\theta \ d\phi \\
=& 4 \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/2} \sin^2\phi \cos\theta d\phi \ d\theta = \pi
\end{aligned}
$$

Volume of solids of revolution: Consider a function $f(x) \geq 0, a \leq x \leq b$ and revolve it around x-axis. Then the region can be represented as

$$R = \{(x, y, z) : a \leq x \leq b, \sqrt{y^2 + z^2} \leq f(x)\}$$

Taking the cylindrical coordinates $x = x, y = r\sin\theta, z = r\cos\theta$, we get

$$R = \{(x, r, \theta) : a \leq x \leq b,\ 0 \leq r \leq f(x), 0 \leq \theta \leq 2\pi\}$$

Therefore, the voloume is

$$
\begin{aligned}
V &= \int_a^b \int_0^{2\pi} \int_0^{f(x)} r dr d\theta dx \\
&= 2\pi \int_a^b \frac{[f(x)]^2}{2} = \pi \int_a^b (f(x))^2 dx
\end{aligned}
$$

Parametrizations of Surfaces: Let

$$\mathbf{r}(u, v) = f(u, v)\hat{i} + g(u, v)\hat{j} + h(u, v)\hat{k}$$

be a continuous vector function defined on a plane region R. The variable u and v are parameters and R is the parameter domain. The range of r is called the surface S. We assume that r is one-to-one on the interior of R so that S does not cross itself.

Examples 1.2.3 *1. A parametrization of the cone*

$$z = \sqrt{x^2 + y^2},\ \ 0 \leq z \leq 1$$

Here cylindrical coordinates provide everything we need. Let

$$x(r, \theta) = r\cos\theta,\ y(r, \theta) = r\sin\theta \ z = \sqrt{x^2 + y^2} = r$$

the domain of r, θ *is* $0 \leq r \leq 1$ *and* $0 \leq \theta \leq 2\pi$*. So the required parametrization is*

$$\mathbf{r}(r, \theta) = r\cos\theta \ \hat{i} + r\sin\theta \ \hat{j} + r \ \hat{k},\ 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi.$$

2. *A parametrization of sphere*

$$x^2 + y^2 + z^2 = a^2$$

Here we take spherical coordinates. Let $x = a\sin\phi\cos\theta, y = a\sin\phi\sin\theta$*, and* $z = a\cos\phi$*,* $0 \le \phi \le \pi, 0 \le \theta \le 2\pi$*. Taking* $u = \phi$ *and* $v = \theta$*, we get*

$$\mathbf{r}(\theta,\phi) = a\sin\phi\cos\theta\ \hat{i} + a\sin\phi\sin\theta\ \hat{j} + a\cos\phi\ \hat{k}, \quad 0 \le \phi \le \pi, 0 \le \theta \le 2\pi.$$

3. *A parametrization of cylinder* $x^2 + (y-3)^2 = 9,\ 0 \le z \le 5$
we take cylindrical coordinates,

$$x(r,\theta) = r\cos\theta,\ y(r,\theta) = r\sin\theta\ z = z$$

Substituting in $x^2 + (y-3)^2 = 9$ *we get* $r^2 + 6r\cos\theta = 0$*. Therefore,* $r = 6\sin\theta$*.*

$$\mathbf{r}(\theta, z) = 3\sin 2\theta\ \hat{i} + 6\sin^2\theta\ \hat{j} + z\ \hat{k},\ 0 \le \theta\pi, 0 \le z \le 5$$