Lecture 39

1 Surface Area and Surface integrals

1.1 Surface Area

Consider a surface S defined with $f(x, y, z) = c$. Let R be its projection on xy-plane. Assume that this projection is **one-one, onto**. Let R_k be a small rectangle with area ΔA_k and let $\Delta\sigma_k$ be the piece of surface above this rectangle. Let ΔP_k be the tangent plane at (x_k, y_k, z_k) of the surface $\Delta\sigma_k$. Now consider the parallelogram with ΔP_k and ΔA_k as upper and lower planes of the parallelogram. We approximate the area of the surface with the area of the tangent plane ΔP_k.

Now let $\hat{p}$ be the unit normal to the plane containing R_k and ∇f is the normal to the surface. Let u_k, v_k be the vectors along the sides of the tangent plane ΔP_k. Then the area of ΔP_k is $|u_k \times v_k|$ and $u_k \times v_k$ is the normal vector to ΔP_k. Thus ∇f and $u_k \times v_k$ are both normals to the tangent plane ΔP_k.

The angle between the plane ΔA_k and ΔP_k is same as the angle between their normals. i.e., the angle between $\hat{p}$ and $u_k \times v_k$. From the geometry, the area of the projection of this tangent plane is $|(u_k \times v_k) \cdot \hat{p}|$(proof of this can be seen in Thomas calculus Appendix 8). i.e.,

$$\Delta A_k = |(u_k \times v_k) \cdot \hat{p}| = |(u_k \times v_k)||\hat{p}|| \cos(\text{angle between } (u_k \times v_k) \text{ and } \hat{p})$$

In other words,

$$\Delta P_k | \cos \gamma_k| = \Delta A_k \quad or \ \Delta P_k = \frac{\Delta A_k}{|\cos \gamma_k|}$$

where γ_k = angle between $(u_k \times v_k)$ and $\hat{p}$. This angle can be calculated easily by noting that ∇f and $u_k \times v_k$ are both normals to the tangent plane.

(This formula is simple in case of straight lines: Let OP be the line from origin and let R be the projection of P on x-axis. Then $OR = OP \cos\gamma$ where γ is the angle between OP and OR. Now imagine the Area of plane is nothing but "sum" of lengths of lines.)

So

$$|\nabla f \cdot \hat{p}| = |\nabla f||\hat{p}|| \cos \gamma_k|$$

Therefore,

$$\text{Surface Area} \approx \sum_k \Delta P_k = \sum_k \frac{|\nabla f|}{|\nabla f \cdot \hat{p}|} \Delta A_k$$

This sum converges to

$$\text{Surface Area} = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \hat{p}|} dA$$

where R is the projection of S on to one of the planes and $\hat{p}$ is the unit normal to the plane of projection.

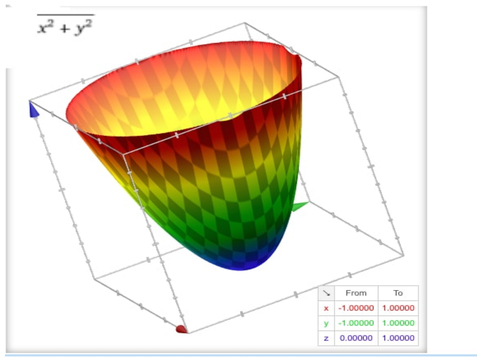

Figure 1: Paraboloid cut by z=2.

Example 1.1.1 *Find the surface area of the curved surface of paraboloid $z = x^2 + y^2$ that is cut by the plane $z = 2$.*

Solution: The equation of surface is $f(x, y, z) = z - x^2 - y^2 = 0$. Clearly this is one-one from xy-plane to $I\!R^3$. So the projection of the surface $\{(x, y, x^2 + y^2) : x^2 + y^2 \leq 2\}$ is the disc $R : x^2 + y^2 \leq 2$. Since the plane of projection is xy-plane, $\hat{p} = \hat{k}$. Hence

$$\nabla f = -2x\hat{\imath} - 2y\hat{\jmath} + \hat{k}$$

$$\begin{aligned} S &= \iint_R \frac{|\nabla f|}{|\nabla f \cdot \hat{p}} dA \\ &= \iint_R \sqrt{4x^2 + 4y^2 + 1} \;\; dA \end{aligned}$$

Going to polar coordinates $x = r\cos\theta, y = r\sin\theta$,

$$S = \int_0^{2\pi} \int_{r=0}^{\sqrt{2}} \sqrt{1 + 4r^2} r \; dr \; d\theta = 13\pi$$

Example 1.1.2 *Find the surface area of the cap obtained by cutting the hemisphere $x^2+y^2+z^2 = 2$ by the cone $z = \sqrt{x^2 + y^2}$.*

Solution: The equation of surface is $f(x, y, z) = x^2 + y^2 + z^2 - 2 = 0$ and we can take the projection onto xy-plane. So $\hat{p} = \hat{k}$. The projection is obtained by solving $x^2 + y^2 + z^2 = 2, z = \sqrt{x^2 + y^2}$. i.e., $R = x^2 + y^2 = 1$.

$$\nabla f = 2x\hat{\imath} + 2y\hat{\jmath} + 2z\hat{k}$$

$$|\nabla f \cdot \hat{p}| = 2z = 2\sqrt{2 - x^2 - y^2}$$

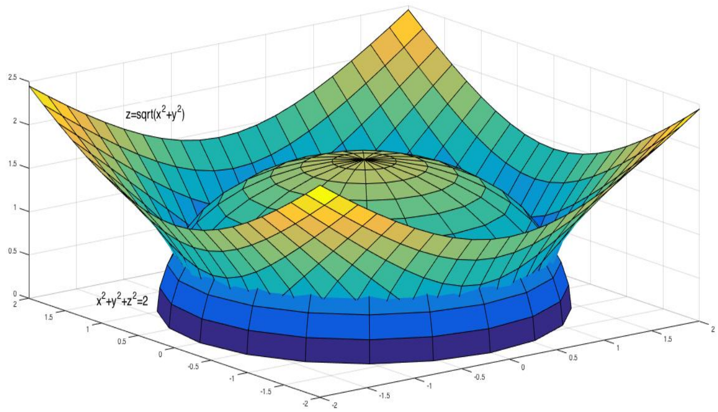

Figure 2: Cap obtained by cutting the hemisphere by the cone.

Therefore, using polar coordinates $x = r\cos\theta, y = r\sin\theta$,

$$\begin{aligned} S &= \iint_R \frac{\sqrt{2}}{\sqrt{2-x^2-y^2}} dA \\ &= \int_0^{2\pi}\int_0^1 \frac{\sqrt{2}}{\sqrt{2-r^2}} r\ dr\ d\theta = 2\pi(2-\sqrt{2}). \end{aligned}$$

Surface area of solids of revolution: If a surface is generated by revolving a curve $z = f(y), y \in (0, \rho)$ in the yz-plane about the z-axis. This surface is the graph of function $z = f(\sqrt{x^2+y^2})$. Then by taking $g(x, y, z) = z - f(\sqrt{x^2+y^2})$ over the circular domain R which is the projection of the solid. So, we get

$$S = \iint_R \frac{|\nabla g|}{|\nabla g \cdot k|} dxdy = \iint_R (1 + f'(\sqrt{x^2+y^2}))dxdy$$

Now using poloar coordinates $x = r\cos\theta, y = r\sin\theta$, we get

$$S = \int_0^{2\pi}\int_0^{\rho} \sqrt{1+(f'(r))^2}\ rdrd\theta = 2\pi\int_{r=0}^{\rho} \sqrt{1+(f'(r))^2}\ rdr$$

1.2 Surface Integrals

Let $g(x, y, z)$ be a function defined over a surface S. Then we can think of integration of g over S. Suppose, a surface S is heated up, we have a temperature distributed over this surface. Let $T(x, y, z)$ be the temperature at (x, y, z) of the surface. Then we can calculate the total temperature on S using the Riemann integration.

Let R be the projection of S on the plane. We partition R into small rectangles A_k. Let ΔS_k be the surface above the ΔA_k. We approximate this surface area element with its tangent plane ΔP_k. As we refine the rectangular partition this ΔP_K approximated the ΔS_k. Then the total temperature may be approximated as

$$\sum_k g(x_k, y_k, z_k)\Delta P_k = \sum_k g(x_k, y_k, z_k)\frac{\Delta A_k}{|\cos\gamma_k|} = \sum_k g(x_k, y_k, z_k)\frac{|\nabla f|}{|\nabla f \cdot \hat{p}|}dA$$

where $\hat{p}$ is the unit normal to R or the plane of projection. Now taking limit $n \to \infty$, we get

$$\iint_S g(x,y,z)dS = \iint_R g(x,y,z)\frac{|\nabla f|}{|\nabla f \cdot \hat{p}|}dA$$

If the surface is defined as $f = z - h(x,y) = 0$, then

$$\iint_S g(x,y,z)dS = \iint_R g(x,y,h(x,y))\frac{|\nabla f|}{|\nabla f \cdot \hat{k}|}dA.$$

Example 1.2.1 *Integrate $g(x,y,z) = z$ over the surface S cut from the cylinder $y^2 + z^2 = 1, z \geq 0$, by the planes $x = 0$ and $x = 1$.*

Solution: $f = y^2 + z^2$ and this surface can be projected 1-1, onto to R of xy plane. This projection is the rectangle with vertices $(1,-1),(1,1),(0,1),(0,-1)$. So $\hat{p} = \hat{k}$

$$\frac{|\nabla f|}{|\nabla f \cdot \hat{p}|} = \frac{2\sqrt{y^2+z^2}}{|2z|} = \frac{1}{z}$$

Therefore,

$$\iint_S z dS = \iint_R z\frac{1}{z}\ dA = Area(R) = 2$$

Parametrizations of Surfaces: Let

$$\mathbf{r}(u,v) = f(u,v)\hat{\imath} + g(u,v)\hat{\jmath} + h(u,v)\hat{k}$$

be a parametrized surface.

Definition 1.2.2 *(Smooth surface): A parametrized surface $\mathbf{r}(u,v)$ is called smooth surface if $\mathbf{r}_u$ and $\mathbf{r}_v$ are continuous and $\mathbf{r}_u \times \mathbf{r}_v$ is never zero.*

Surface Area: We approximate the surface area element by the parallelogram on the tangent plane whose sides are determined by the vectors $r_u\Delta u$ and $r_v\Delta v$. The total surface area is approximately equal to the sum of area of of this parallelograms

$$S \sim \sum_u \sum_v |r_u \times r_v|\Delta u\Delta v$$

This sum is a Riemann sum of the integral $\int_a^b \int_a^b |r_u \times r_v|\ dudv$. Therefore we have
The Surface area of smooth surface: $\mathbf{r}(u,v) = f(u,v)\hat{\imath} + g(u,v)\hat{\jmath} + h(u,v)\hat{k},\ a \leq u \leq b,\ c \leq v \leq d$ is

$$S = \int_a^b \int_c^d |\mathbf{r}_u \times \mathbf{r}_v|\ dudv$$

Problem 1.2.1 *Find the surface are of the surface of the cone*

$$z = \sqrt{x^2 + y^2}, \;\; 0 \le z \le 1$$

Solution: We found a parametrization of the cone as

$$\mathbf{r}(r, \theta) = r\cos\theta \; \hat{i} + r\sin\theta \; \hat{j} + r \; \hat{k}, \; 0 \le r \le 1, 0 \le \theta \le 2\pi.$$

We can find that

$$|\mathbf{r}_r \times \mathbf{r}_\theta| \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2} = \sqrt{2}r$$

Therefore,

$$\text{Surface Area} = \int_0^{2\pi}\int_0^1 \sqrt{2} r dr d\theta = \pi\sqrt{2}$$

Surface integrals: Let $F(u, v)$ be a continuous function defined on the parametrized surface S: $\mathbf{r}(u, v) : R \to S$, where $R : a \le u \le b, c \le v \le d$. Then

$$\iint_S F dS = \int_a^b \int_c^d F(u, v)|\mathbf{r}_u \times \mathbf{r}_v| du dv$$

Problem 1.2.2 *Evaluate the surface integral* $\iint_S (x + y + z)dS$ *over the surface of cylinder* $x^2 + y^2 = 9, 0 \le z \le 4$.

Solution: Using the cylindrical coordinates: $x = 3\cos\theta, y = 3\sin\theta, z = z$ over the parameter domain $\{(\theta, z) : 0 \le \theta \le 2\pi, 0 \le z \le 4\}$. The surface can be represented as

$$T(\theta, z) = 3\cos\theta\hat{i} + 3\sin\theta\hat{j} + z\hat{k}.$$

Then $|T_\theta \times T_z| = \sqrt{9\cos^2\theta + 9\sin^2\theta} = 3$. The given integral is equal to

$$\iint_S (x + y + z)dS = \iint_S (3\cos\theta + 3\sin\theta + z)|T_\theta \times T_z| d\theta dz$$
$$\int_{z=0}^4 \int_{\theta=0}^{2\pi} (3\cos\theta + 3\sin\theta + z) d\theta dz = 6\pi \int_0^4 z dz = 48\pi.$$