Lecture 4

1 Bolzano-Weierstrass Theorem

1.1 Divergent sequence and Monotone sequences

Definition 1.1.1. *Let $\{a_n\}$be a sequence of real numbers. We say that a_n approaches infinity or diverges to infinity, if for any real number $M > 0$, there is a positive integer N such that*

$$n \geq N \implies a_n \geq M.$$

- If a_n approaches infinity, then we write $a_n \to \infty$ as $n \to \infty$.
- A similar definition is given for the sequences diverging to $-\infty$. In this case we write $a_n \to -\infty$ as $n \to \infty$.

Examples 1.1.2.

(i) The sequence $\{\log(1/n)\}_1^\infty$ diverges to $-\infty$. In order to prove this, for any $M > 0$, we must produce a $N \in \mathbb{N}$ such that

$$\log(1/n) < -M, \quad \forall\, n \geq N.$$

But this is equivalent to saying that $n > e^M$, $\forall\, n \geq N$. Choose $N \geq e^M$. Then, for this choice of N,

$$\log(1/n) < -M, \quad \forall\, n \geq N.$$

Thus $\{\log(1/n)\}_1^\infty$ diverges to $-\infty$.

Definition 1.1.3. *If a sequence $\{a_n\}$does not converge to a value in $\mathbb{R}$ and also does not diverge to ∞ or $-\infty$, we say that $\{a_n\}$oscillates.*

Theorem 1.1.4. *Let $\{a_n\}$and $\{b_n\}$ be two sequences.*

(i) If $\{a_n\}$and $\{b_n\}$ both diverges to ∞, then the sequences $\{a_n + b_n\}$ and $\{a_n b_n\}$ also diverges to ∞.

(ii) If $\{a_n\}$diverges to ∞ and $\{b_n\}$ converges then $\{a_n + b_n\}$ diverges to ∞.

Example 1.1.5. *Consider the sequence $\{\sqrt{n+1} - \sqrt{n}\}_{n=1}^\infty$. We know that $\sqrt{n+1}$ and $\sqrt{n}$ both diverges to ∞. But the sequence $\{\sqrt{n+1} - \sqrt{n}\}_{n=1}^\infty$ converges to 0. To see this, notice that, for a given $\epsilon > 0$, $\sqrt{n+1} - \sqrt{n} < \epsilon$ if and only if $1 < \epsilon^2 + 2\epsilon\sqrt{n}$. Thus, if N is such that $N > \dfrac{1}{4\epsilon^2}$, then for all $n \geq N$, $\sqrt{n+1} - \sqrt{n} < \epsilon$. Thus $\sqrt{n+1} - \sqrt{n}$ converges to 0. This example shows that the sequence formed by taking difference of two diverging sequences may converge.*

Definition 1.1.6. Monotone sequence
A sequence $\{a_n\}$ of real numbers is called a nondecreasing sequence if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$

and $\{a_n\}$ *is called a nonincreasing sequence if* $a_n \geq a_{n+1}$ *for all* $n \in \mathbb{N}$. *A sequence that is nondecreasing or nonincreasing is called a monotone sequence.*

Examples 1.1.7.

(i) *The sequences* $\{1 - 1/n\}$, $\{n^3\}$ *are nondecreasing sequences.*

(ii) *The sequences* $\{1/n\}$, $\{1/n^2\}$ *are nonincreasing sequences.*

(iii) *The sequences* $\{(-1)^n\}$, $\{cos(\frac{n\pi}{3})\}$, $\{(-1)^n n\}$, $\{\frac{(-1)^n}{n}\}$ *and* $\{n^{1/n}\}$ *are not monotonic sequences.*

Remark 1.1.

(i) *A nondecreasing sequence which is not bounded above diverges to* ∞.

(i) *A nonincreasing sequence which is not bounded below diverges to* $-\infty$.

Example 1.1.8. *If* $b > 1$, *then the sequence* $\{b^n\}_1^\infty$ *diverges to* ∞.

Theorem 1.1.9.

(i) *A nondecreasing sequence which is bounded above is convergent.*

(ii) *A nonincreasing sequence which is bounded below is convergent.*

Proof. (i) Let $\{a_n\}$be a nondecreasing, bounded above sequence and $a = \sup\limits_{n \in \mathbb{N}} a_n$. Since the sequence is bounded, $a \in \mathbb{R}$. We claim that a is the limit point of the sequence $\{a_n\}$. Indeed, let $\epsilon > 0$ be given. Since $a - \epsilon$ is not an upper bound for $\{a_n\}$, there exists $N \in \mathbb{N}$ such that $a_N > a - \epsilon$. As the sequence is nondecreasing, we have $a - \epsilon < a_N \leq a_n$ for all $n \geq N$. Also it is clear that $a_n \leq a$ for all $n \in \mathbb{N}$. Thus,

$$a - \epsilon \leq a_n \leq a + \epsilon, \ \forall\, n \geq N.$$

Hence the proof.

The proof of (ii) is similar to (i) and is left as an exercise to the students. ///

Examples 1.1.10.

(i) *If* $0 < b < 1$, *then the sequence* $\{b^n\}_1^\infty$ *converges to* 0.

Solution. *We may write* $b^{n+1} = b^n b < b^n$. *Hence* $\{b^n\}$ *is nonincreasing. Since* $b^n > 0$ *for all* $n \in \mathbb{N}$, *the sequence* $\{b^n\}$ *is bounded below. Hence, by the above theorem,* $\{b^n\}$ *converges. Let* $L = \lim\limits_{n\to\infty} b^n$. *Further,* $\lim\limits_{n\to\infty} b^{n+1} = \lim\limits_{n\to\infty} b \cdot b^n = b \cdot \lim\limits_{n\to\infty} b^n = b \cdot L$. *Thus the sequence* $\{b^{n+1}\}$ *converges to* $b \cdot L$. *On the other hand,* $\{b^{n+1}\}$ *is a subsequence of* $\{b^n\}$. *Hence* $L = b \cdot L$ *which implies* $L = 0$ *as* $b \neq 1$.

(ii) *The sequence* $\{(1 + 1/n)^n\}_1^\infty$ *is convergent.*

Solution. *Let* $a_n = (1+1/n)^n = \sum_{k=0}^{n} \binom{n}{k}\left(\frac{1}{n}\right)^k$. *For* $k = 1, 2, ..., n$, *the* $(k+1)^{th}$ *term in the expansion is*

$$\frac{n(n-1)(n-2)\cdots(n-k+1)}{1\cdot 2\cdots k}\frac{1}{n^k} = \frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right). \tag{1.1}$$

Similarly, if we expand a_{n+1}, *then we obtain* $(n+2)$ *terms in the expansion and for* $k = 1, 2, 3, ...,$ *the* $(k+1)^{th}$ *term is*

$$\frac{1}{k!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{k-1}{n+1}\right) < \frac{1}{k!}. \tag{1.2}$$

It is clear that (1.2) is greater than or equal to (1.1) and hence $a_n \le a_{n+1}$ *which implies that* $\{a_n\}$*is nondecreasing. Further,*

$$a_n = (1+1/n)^n = \sum_{k=0}^{n} \binom{n}{k}\left(\frac{1}{n}\right)^k < 1 + \sum_{k=1}^{n}\frac{1}{k!} < 1 + 2 = 3.$$

$\left(k! > 2^{k-1} \implies \sum_{k=1}^{n}\frac{1}{k!} < \sum_{k=1}^{n}\frac{1}{2^{k-1}} < 2\right)$ *for each* n. *Thus* $\{a_n\}$*is a bounded monotone sequence and hence convergent.*

Theorem 1.1.11. *Every sequence has a monotone subsequence.*

Proof. Pick x_{N_1} such that $x_n \le x_{N_1}$ for all $n > N_1$. We call such x_N as "peak". If we are able to pick infinitely many $x'_{N_i}s$, then $\{x_{N_i}\}$ is decreasing and we are done. If there are only finitely many $x'_N s$ and let x_{n_1} be the last peak. Then for $n_2 > n_1$, x_{n_2} is not a peak. That means we can choose n_3 such that $x_{n_3} \ge x_{n_2}$. Again x_{n_3} is not a peak. So we can choose x_{n_4} such that $x_{n_4} \ge x_{n_3}$. Proceeding this way, we get a non-decreasing sub-sequence $\{x_{n_2}, x_{n_3}, x_{n_4}, ...\}$. ///
The following theorem is Bolzano-Weierstrass theorem. Proof is a consequence of Theorem1.1.11

Theorem 1.1.12. *Every bounded sequence has a convergent subsequence.*

Theorem 1.1.13. *Nested Interval theorem: Let* $I_n = [a_n, b_n], n \ge 1$ *be non-empty closed, bounded intervals such that*

$$I_1 \supset I_2 \supset I_3 ... \supset I_n \supset I_{n+1} ...$$

and $\lim_{n\to\infty}(b_n - a_n) = 0$. *Then* $\cap_{n=1}^{\infty} I_n$ *contains precisely one point*

Proof. Since $\{a_n\}, \{b_n\} \subset [a_1, b_1]$, $\{a_n\}, \{b_n\}$ are bounded sequences. By Bolzano-Weierstrass theorem, there exists sub sequences a_{n_k}, b_{n_k} and a, b such that $a_{n_k} \to a, b_{n_k} \to b$. Since a_n is increasing $a_1 < a_2 < ...... \le a$ and $b_1 > b_2 > .... \ge b$. It is easy to see that $a \le b$. Also since $0 = \lim a_n - b_n = a - b$, we have $a = b$.
It is easy to show that there is no other point in $\cap_{n=1}^{\infty} I_n$. ///

Remark 1.2. *closedness of* I_n *cannot be dropped. for example the sequence* $\{(0, \frac{1}{n})\}$. *Then* $\cap_{n=1}^{\infty}(0, \frac{1}{n}) = \emptyset$ *because there cannot be any element* x *such that* $0 < x < \frac{1}{n}$ *else Archimedean property fails.*

Corollary 1.1.14. $\mathbb{R}$ *is uncountable.*

Proof. It is enough to show that $[0, 1]$ is uncountable. If not, there exists an onto map $f : \mathbb{N} \to [0, 1]$. Now subdivide $[0, 1]$ into 3 equal parts so that choose J_1 such that $f(1) \notin J_1$. Now subdivide J_1 into 3 equal parts and choose J_2 so that $f(2) \notin J_2$. Continue the process to obtain J_n so that $f(n) \notin J_n$. These J_n satisfy the hypothesis of above theorem, so $\cap_{n=1}^{\infty} = \{x\}$ and $x \in [0, 1]$. By the construction, there is no $n \in \mathbb{N}$ such that $f(n) = x$. contradiction to f is onto.