Lecture 40

1 Vector Calculus

1.1 Line integrals and Green's theorem

In many physical phenomena, the integrals over paths through vector filed plays important role. For example, work done in moving an object along a path against a variable force or to find work done by a vector field in moving an object along a path through the field. A **vector field** on a domain in the plane or in the space is a vector valued function $f : \mathbb{R}^3 \to \mathbb{R}^3$ with components say M, N and P, for example

$$F(x, y, z) = M(x, y, z)\hat{\imath} + N(x, y, z)\hat{\jmath} + P(x, y, z)\hat{k}$$

We assume that M, N, P are continuous functions. Suppose F represents a force throughout a region in space and let $r(t) = x(t)\hat{\imath}+y(t)\hat{\jmath}+z(t)\hat{k}, a \leq t \leq b$ is a smooth curve in the region. Then we introduce the partition $a = t_1 < t_2 ..... < t_n = b$ of $[a, b]$.

If F_k denotes the value of F at the point on the corresponding to t_k on the curve and T_k denotes the curve's unit tangent vector a this point. Then $F_k \cdot T_k$ is the scalar component of F in the director of T at t_k. Then the work done by F along the curve is approximately

$$\sum_{k=1}^{n} F_k \cdot T_k \Delta s_k,$$

where Δs_k is the length of the curve between t_{k-1}, t_k. As the norm of the partition approaches zero, these sum's approaches

$$\int_{t=a}^{b} F \cdot T ds = \int_a^b \overrightarrow{F} \cdot \overrightarrow{T} \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt.$$

Now substituting $T(t) = \frac{\overrightarrow{r}'(t)}{|\overrightarrow{r}'(t)|}$, we get

$$\int_a^b \overrightarrow{F} \cdot \overrightarrow{r}'(t) dt.$$

The following can be shown:

Theorem 1.1.1 *The Line integral is independent of choice of parametrization.*

Definition 1.1.1 *The* **orientation** *of a parameterized curve is the direction determined by increasing values of the parameter.*

The line integral over a parametrized curve depends on the orientation. If we change the orientation, then the integral is equal to -1 times: That is,

$$-\int_{\mathcal{C}} F \cdot dr = \int_{-\mathcal{C}} F \cdot dr.$$

Example 1.1.2 *Find the work done by* $F = 3x^2\hat{i} + (2xz - y)\hat{j} - z\hat{k}$ *over the curve* $r(t) = t\hat{i} + t^2\hat{j} + t^3\hat{k}, 0 \leq t \leq 1$ *from origin to* $(1, 1, 1)$

Solution: The tangent along the curve T is $\frac{dr}{dt}$. Therefore,

$$\begin{aligned}\int_0^1 F \cdot T ds &= \int_0^1 F \cdot \frac{d\overrightarrow{r}}{dt} dt \\ &= \int_0^1 3t^2 + t^5 - 2t^3 dt = \frac{2}{3}.\end{aligned}$$

Definition 1.1.3 *Conservative vector field: A verctor field* $\overrightarrow{F}$ *is called conservative vector field if the line integral depends only on the end points. Equivalently, the line integral over any closed curve is zero.*

The following is knon as fundamental thoerem of Line integrals. Recall the fundamental theorem of integral calculus.

Theorem 1.1.2 *Let* $\overrightarrow{F}$ *be a vector field and if there exists a differentiable function* $f : I\!R^3 \to I\!R$ *such that* $\overrightarrow{F} = \nabla f$. *Then*

$$\int_P^Q \overrightarrow{F} \cdot d\overrightarrow{r} = f(Q) - f(P)$$

Proof. Suppose there exists f such that $\overrightarrow{F} = \nabla f$. let $\overrightarrow{r}(t) : \{(x(t), y(t), z(t)), t \in [0, 1]\}$ represent a curve connecting P and Q. Then

$$\begin{aligned}\int_P^Q \overrightarrow{F} \cdot d\overrightarrow{r} &= \int_0^1 \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \right) dt \\ &= \int_0^1 \frac{d}{dt}(f(x(t), y(t), z(t)))dt \\ &= f(Q) - f(P).\end{aligned}$$

It is clear that if $P = Q$, then above integral is zero.

Theorem 1.1.3 *Let* $\overrightarrow{F} = M\hat{i} + N\hat{j} + P\hat{k}$ *be a conservative whose components are continuous over an open connected domain* D *in* $\mathbb{R}^3$. *Then there exists a differential function* f *such that* $\overrightarrow{F} = \nabla f$.

Proof. let X_0 be a fixed point in D. For any point (x, y, z) in D, let C be a path from X_0 to (x, y, z). Define $f(x, y, z)$ by

$$f(x, y, z) = \int_C \overrightarrow{F} \cdot d\overrightarrow{r}.$$

(Note that this definition of f makes sense only because it is independent of path). We want to show that f satisfies $\nabla f = \overrightarrow{F}$.

Since domain D is open, it is possible to find a disk centered at (x, y, z) such that the disk is contained entirely inside D. Let (a, y, z) with $a < x$ be a point in that disk. Let C be a path from X_0 to (x, y, z) that consists of two pieces: C_1 and C_2. The first piece, C_1, is any path from C to (a, y, z) that stays inside D and C_2 is the horizontal line segment from (a, y, z) to (x, y, z). Then

$$f(x,y,z) = \int_{C_1} \overrightarrow{F} \cdot d\overrightarrow{r} + \int_{C_2} \overrightarrow{F} \cdot d\overrightarrow{r}$$

Then differentiating this with respect to x, we see that the $\frac{\partial}{\partial x}$ of the first term on the right hand side is zero. Therefore

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \int_{C_2} \overrightarrow{F} \cdot d\overrightarrow{r}$$

Now considering the parametrization $\overrightarrow{r}(t) = t\hat{i} + y\hat{j} + z\hat{k}$, $a \le t \le x$. Then $\overrightarrow{F} \cdot d\overrightarrow{r} = M(t, y, z)$. Hence

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \int_a^x M(t,y,z)dt = M(x,y,z)$$

thanks to fundamental theorem of integral calculus. A similar argument using a lines parallel to y-axis and z-axis rather than a line parallel to $x-$axis, shows that $f_y = N(x, y, z)$ and $f_y = N(x, y, z)$. □

Divergence and Curl: For a vector field $\overrightarrow{F} = M\hat{i} + N\hat{j} + P\hat{k}$ the Divergence and curl are defined as

$$div\overrightarrow{F} = \nabla \cdot \overrightarrow{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

$$curl\overrightarrow{F} = \nabla \times \overrightarrow{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

The following theorem is known as curl-div theorem:

Theorem 1.1.4 *Suppose $\overrightarrow{F}$ is a vector field with all its components have continuous second order partial derivaties then*

$$div(curl\overrightarrow{F}) = 0.$$

Proof. Proof is a simple calculation. □

The following is a neccessary condition.

Theorem 1.1.5 *Suppose F is a onservative vector field with all its components are differentiable and partial derivatives are continuous in the domain D. Then $curl\overrightarrow{F} = 0$.*

Proof. Since $\overrightarrow{F}$ is conservative, there exists f such that $\overrightarrow{F} = \nabla f$. Therefore, second order partial derivatives are continuous. Hence mixed derivatives are equal. Now it is easy to check that $\nabla \times \nabla f = 0$. Therefore $curl\overrightarrow{F} = 0$. □

Definition 1.1.4 *A vector field is irrotational if* $curl\overrightarrow{F} = 0$.

Now we ask the question: Is irrotational vector field is always conservative?
The answer is NO. The following example

Example 1.1.5 *Let* $\overrightarrow{F} = F_1\hat{i} + F_2\hat{j} = -\frac{y}{x^2+y^2}\hat{i} + \frac{x}{x^2+y^2}\hat{j}$. *Then* $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$ *implying* $curl\overrightarrow{F} = 0$. *But the line integral* $\int_C \overrightarrow{F} \cdot d\overrightarrow{r} = \int_0^{2\pi} \sin^2\theta + \cos^2\theta = 2\pi$.

But depending on the shape of the domain D, $curl\overrightarrow{F} = 0$ implies $\overrightarrow{F}$ is conservative.

Definition 1.1.6 *A subset* D *of* $\mathbb{R}^n$ *is called simply connected if it is path-connected and and every loop in* D *can be contracted to a point without ever leaving the domain.*

Examples 1.1.7

1. *The whole space* $\mathbb{R}^n$ *is simply connected.*
2. *The unit ball* $\{(x,y,z) : x^2+y^2+z^2 < 1\}$ *is simply connected.*
3. *The annulus* $\{(x,y,z) : 1 < x^2+y^2+z^2 < 2\}$ *is NOT connected.*
4. *The punctured disc* $\{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1\}\backslash\{(0,0)\}$ *is NOT simply connected.*
5. *The punctured space* $\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 < 1\}\backslash\{(0,0,0)\}$ *is simply connected.*
6. $\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 < 1\}\backslash\{z - axis\}$ *is NOT simply connected.*

Theorem 1.1.6 *Let* $\overrightarrow{F}$ *be a vector field with its components are differentiable and partial derivatives are continuous. Then* $curl\overrightarrow{F} = 0$ *implies* $\overrightarrow{F}$ *is conservative.*

Proof. Proof requires Stoke's theorem. We omit it.

Example 1.1.8 *Consider the vector field* $\overrightarrow{F} = (3x^2y^2z + 5y^3)\hat{i} + (2x^3yz + 15xy^2 - 7z)\hat{j} + (x^3y^2 - 7y + 4z^3)\hat{k}$ *Determine whether* $\overrightarrow{F}$ *is conservative, and if it is, find a potential function* f *for which* $\overrightarrow{F} = \nabla f$.

Solution It is easy to check that $curl\overrightarrow{F} = 0$ and domain of definition of $\overrightarrow{F}$ is the whole space $\mathbb{R}^3$ which is simply connnected. Therefore, $\overrightarrow{F}$ is conservative.
To find the function f, we take

$$f_x = 3x^2y^2z + 5y^3,\ f_y = 2x^3yz + 15xy^2 - 7z,\ f_z = x^3y^2 - 7y + 4z^3$$

Using the first equation we obtain

$$f(x,y,z) = \int (3x^2y^2z + 5y^3)dx = x^3y^2z + 5y^3x + g(y,z)$$

Differentiation with respect to y and equating with second equation we get

$$2x^3yz + 15y^2x + g_y(y,z) = 2x^3yz + 15xy^2 - 7z \implies g_y = -7z$$

Integrating g_y we get $g(y,z) = -7yz + h(z)$. Hence $f = x^3y^2z + 5y^3x - 7yz + h(z)$. Substituting this the third equation, we get $h'(z) = 4z^3$. Therefore $h(z) = z^4 + c$. Hence $f(x,y,z) = x^3y^2z + 5y^3x - 7yz + z^4 + c$. □

Example 1.1.9 *Consider the vector field* $\overrightarrow{F} = \frac{1}{x^2+y^2+z^2}(x\hat{\imath} + y\hat{\jmath} + z\hat{k})$*. Determine whether* $\overrightarrow{F}$ *is conservative, and if it is, find a potential function* f *for which* $\overrightarrow{F} = \nabla f$*.*

Solution: In this case again the domain of definition is the punctured space $\mathbb{R}^3 \backslash \{(0,0,0)\}$ which is simply connected. Also it is easy to check

$$\nabla \times \overrightarrow{F} = \frac{-2}{(x^2+y^2+z^2)^2}\Big((zy - yz)\hat{\imath} + (xz - zx)\hat{\jmath} + (yx - xy)\hat{k}\Big) = 0.$$

Now following the steps as above, it is not difficult to find f as

$$f(x,y,z) = \frac{1}{2}\ln(x^2+y^2+z^2).$$