Lecture 41

1 Integral theorems

1.1 Green's theorem in the plane

Theorem 1.1.1 *Let R be a closed bounded region in $\mathbb{R}^2$ whose boundary $\mathcal{C}$ consists of finitely many smooth curves. Let $\overrightarrow{F}(x,y) = F_1(x,y)\hat{\imath} + F_2(x,y)\hat{\jmath}$ be continuous and has continuous partial derivatives everywhere in some domain containing R. Then*

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA = \oint_C \overrightarrow{F} \cdot d\overrightarrow{r},$$

where the line integral is along the boundary $\mathcal{C}$ of R such that R is on the left as we advance on the boundary. □

Proof. We omit the proof. A proof in special case can be found in the reference/text book. □

Example 1.1.1 *Evaluate $\int_C \overrightarrow{F} \cdot d\overrightarrow{r}$ for $\overrightarrow{F} = (y^2 - 7y)\hat{\imath} + (2xy + 2x)\hat{\jmath}$ and $C : x^2 + y^2 = 1$.*

Solution:

$$\int_C \overrightarrow{F} \cdot d\overrightarrow{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA = 9 \iint_R dA = 9\pi.$$

□

Greens theorem can be applied to non simply connected domains like annular regions for example

Example 1.1.2 *Let R be the domain $\{(x,y) : 1 < x^2 + y^2 < 2\}$ and let C be the (positively oriented) boundary of the domain R. Then evaluate $\int_C \frac{ydx - xdy}{x^2+y^2}$.*

Solution: Boundary of R consists of two circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 2$. Therefore Green's theorem can be applied. Take $\overrightarrow{F} = F_1\hat{\imath} + F_2\hat{\jmath} = \frac{1}{x^2+y^2}(y\hat{\imath} - x\hat{\jmath})$. Then

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0 \implies \iint_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} dA = 0.$$

Therefore by Green's theorem

$$\int_C \frac{ydx - xdy}{x^2 + y^2} = \int_C \overrightarrow{F} \cdot d\overrightarrow{r} = \iint_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} dA = 0.$$

However the following we give an important example

Example 1.1.3 *Let the region R be the punctured disc $\{(x,y) : x^2+y^2 \leq 1\}\backslash\{0\}$. and let $\overrightarrow{F} = -\dfrac{y}{x^2+y^2}\hat{i} + \dfrac{x}{x^2+y^2}\hat{j}$. Then $\dfrac{\partial F_2}{\partial x} = \dfrac{\partial F_1}{\partial y}$ and $\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) = 0$. But the line integral $\displaystyle\int_C \overrightarrow{F} \cdot d\overrightarrow{r} = \int_0^{2\pi} \sin^2\theta + \cos^2\theta = 2\pi$. (taking the parametrization $x = \cos\theta, y = \sin\theta$, 0 varies from 0 to 2π)*

Another application of the Green's theorem is

Problem 1.1.1 *Let C be the closed curve defined as $C = C_1 + C_2$ where $C_1 : y + |x| = 2,\ 0 \leq y \leq 2,\ \ -2 \leq x \leq 2$ and $C_2 : x^2+y^2 = 4,\ \ -2 \leq y \leq 0,\ \ -2 \leq x \leq 2$. Suppose $\overrightarrow{F} = -\frac{y}{x^2+y^2}\hat{i} + \frac{x}{x^2+y^2}\hat{j}$. Then find $\int_C \overrightarrow{F} \cdot d\overrightarrow{r}$.*

Solution: Let R_1 be the region bounded by C. Consider the annural Region R defined as $R_1\backslash B_1(0)$. Note that

$$\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y} \implies \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) = 0.$$

Now by Green's theorem

$$\int_{C\cup\partial B_1(0)} \overrightarrow{F} \cdot d\overrightarrow{r} = 0.$$

Therefore

$$\int_C \overrightarrow{F} \cdot d\overrightarrow{r} = \int_{\partial B_1(0)} \overrightarrow{F} \cdot d\overrightarrow{r} = 2\pi.$$

Area of plane region:

Using Green's theorem, we can write area of a plane region as a line integral over the boundary. Choose $F_1 = 0, F_2 = x$ and then $F_= -y, F_2 = 0$. This gives

$$\iint_R dA = \int_C xdy \text{ and } \iint_R dA = -\int_C ydx$$

respectively. The double integral is the area A of R. By addition we have

$$A = \frac{1}{2}\int_C (xdy - ydx)$$

Example 1.1.4 *Area bounded by ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.*

solution: Take $x = a\cos t, y = b\sin t, 0 \leq t \leq 2\pi$. Then by above formula

$$A = \frac{1}{2}\int_0^{2\pi} (xy' - yx')dt = \frac{1}{2}\left(ab\cos^2 t - (-ab\sin^2 t)\right)dt = \pi ab$$

1.2 Gauss and Stokes theorems

Let S be a smooth surface and we may choose unit normal $\hat{n}$ at P of S. The direction of $\hat{n}$ is called positive normal direction of S at P. We call a smooth surface S orientable surface if the positive normal at P can be continued in a unique and continuous way to the entire surface. For example the Mobius strip is not orientable. A normal at a point P of this strip is displaced continuously along a closed curve C, the resulting normal upon returning to P is opposite to the original vector at P.

Theorem 1.2.1 Gauss Divergence Theorem
Let Ω be a closed, bounded region in $I\!R^3$ whose boundary is a piecewise smooth orientable surface S. Let $\overrightarrow{F}(x, y, z)$ be a continuous function that has continuous partial derivatives in some domain containing Ω. Then

$$\iiint_\Omega \nabla \cdot F dV = \iint_S \overrightarrow{F} \cdot \hat{n} dS$$

where $\hat{n}$ is the outer unit normal vector of S.

Proof. We omit the proof. A proof in special case can be found in the reference/text book. □

Example 1.2.1 *Evaluate $\iint_{\partial\Omega} \overrightarrow{F}.\hat{n} dA$ where $\partial\Omega$ is the boundary of the domain inside the cylinder $x^2 + y^2 = 1$ and between the planes $z = 0, z = x + 2$ and $\overrightarrow{F} = (x^2 + ye^z)\hat{\imath} + (y^2 + ze^2)\hat{\jmath} + (z^2 + xe^y)\hat{k}$.*

Solution: With the given $\overrightarrow{F}$, it is not difficult to obtain, $\nabla \cdot \overrightarrow{F} = 2x+2y+2z$. By Divergence theorem

$$\iint_{\partial\Omega} \overrightarrow{F} \cdot \hat{n} dS = \iiint_\Omega 2(x + y + z)dV = 2 \iint_{x^2+y^2\le 1} \left(\int_{z=0}^{x+2} (x + y + z)dz \right) dxdy$$

Theorem 1.2.2 Stokes's theorem
Let S be a piecewise smooth oriented surface with boundary and let boundary $\mathcal{C}$ be a simple closed curve. Let $\overrightarrow{F}$ be a continuous function which has continuous partial derivatives in a domain containing S. Then

$$\iint_S (\nabla \times \overrightarrow{F}) \cdot \hat{n} dS = \int_{\mathcal{C}} \overrightarrow{F} \cdot d\overrightarrow{r}$$

where $\hat{n}$ is a unit normal vector of S and, depending on $\hat{n}$, the integration around C is taken in the way that S lies in the left of $\mathcal{C}$. Here $\hat{n}$ is the direction of your head while moving along the boundary with surface on your left.

Proof. We omit the proof. A proof in special case can be found in the reference/text book. □

Example 1.2.2 *Evaluate* $\int_C \overrightarrow{F} \cdot d\overrightarrow{r}$ *where* $\overrightarrow{F} = x^2y^3\hat{i} + \hat{j} + z\hat{k}$ *and* C *The intersection of the cylinder* $x^2 + y^2 = 4$ *and the hemisphere* $x^2 + y^2 + z^2 = 16, z \geq 0$.

Solution: The intersection of cylinder and sphere is the boundary of cylinder on the plane $z = \sqrt{12}$. The unit normal to the surface is $\hat{n} = \frac{1}{4}(x\hat{i} + y\hat{j} + z\hat{k})$. The projection R of S on the xy-plane is the disc $x^2 + y^2 \leq 2$, $\nabla \times \overrightarrow{F} = -3x^2y^2\hat{k}$ and $\frac{|\nabla f|}{|\nabla f \cdot \hat{p}|} = \frac{4}{z}$. Hence by Stoke's theorem

$$\begin{aligned}\oint_C \overrightarrow{F} \cdot \overrightarrow{dr} &= \iint_R (-\frac{3}{4})x^2y^2z\frac{4}{z}dA \\ &= -3\int_{\theta=0}^{2\pi}\int_{r=0}^{2}(r^2\cos^2\theta)(r^2\sin^2\theta)rdrd\theta = -8\pi.\end{aligned}$$

Suppose S_1, S_2 be two surfaces having the same boundary curve C. An important consequence of Stoke's theorem is that flux through S_1 or S_2 is same.

Example 1.2.3 *Suppose* S *is a surface of a light bulb over the unit disc* $x^2 + y^2 = 1$ *oriented with outward pointing normal. Suppose* $\overrightarrow{F} = e^{z^2-2z}x\hat{i} + (\sin(xyz) + y + 1)\hat{j} + e^{z^2}\sin(z^2)\hat{k}$. *Compute* $\iint_S (\nabla \times \overrightarrow{F}) \cdot \hat{n}dS$.

Solution: Enough to take any surface with boundary $x^2 + y^2 = 1$. So we take the unit disc $x^2+y^2 \leq 1, z = 0$. Then $\overrightarrow{F}$ on this is $\overrightarrow{F} = x\hat{i}+(y+1)\hat{j}$. Then $\nabla\times\overrightarrow{F} = 0$. Hence $\int_C \overrightarrow{F}\cdot d\overrightarrow{r} = 0$.

References

1. Thomas' Calculus, Chapters 15, 16
2. Advanced engineering Mathematics, E.Kreyszig, Chapter 9