Lecture 5

1 Cauchy sequences

Definition 1.0.1. *A sequence* $\{a_n\}$*is called a Cauchy sequence if for any given* $\epsilon > 0$*, there exists* $N \in \mathbb{N}$ *such that* $n, m \geq N \implies |a_n - a_m| < \epsilon$.

Example 1.0.2. *Let* $\{a_n\}$*be a sequence such that* $\{a_n\}$*converges to* L *(say). Let* $\epsilon > 0$ *be given. Then there exists* $N \in \mathbb{N}$ *such that*

$$|a_n - L| < \frac{\epsilon}{2} \ \forall\, n \geq N.$$

Thus if $n, m \geq N$*, we have*

$$|a_n - a_m| \leq |a_n - L| + |a_m - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus $\{a_n\}$*is Cauchy.*

Lemma 1.0.3. *If* $\{a_n\}$*is a Cauchy sequence, then* $\{a_n\}$*is bounded.*

Proof. Since $\{a_n\}$forms a Cauchy sequence, for $\epsilon = 1$ there exists $N \in \mathbb{N}$ such that

$$|a_n - a_m| < 1, \ \forall\, n, m \geq N.$$

In particular,

$$|a_n - a_N| < 1, \ \forall\, n \geq N.$$

Hence if $n \geq N$, then

$$|a_n| \leq |a_n - a_N| + |a_N| < 1 + |a_N|, \ \forall\, n \geq N.$$

Let $M = \max\{|a_1|, |a_2|, ..., |a_{N-1}|, 1 + |a_N|\}$. Then $|a_n| \leq M$ for all $n \in \mathbb{N}$. Hence $\{a_n\}$is bounded. ///

Theorem 1.0.4. *If* $\{a_n\}$*is a Cauchy sequence, then* $\{a_n\}$*is convergent.*

Proof. Let a_{n_k} be a monotone subsequence of the Cauchy sequence $\{a_n\}$. Then a_{n_k} is a bounded, monotone subsequence. Hence $\{a_{n_k}\}$ converges to L(say). Now we claim that the sequence $\{a_n\}$ itself converges to L. Let $\epsilon > 0$. Choose N_1, N_2 such that

$$n, n_k \geq N_1 \implies |a_n - a_{n_k}| < \epsilon/2$$

$$n_k \geq N_2 \implies |a_{n_k} - a| < \epsilon/2.$$

Then

$$n, n_k \geq \max\{N, N_1\} \implies |a_n - a| \leq |a_n - a_{n_k}| + \| a_{n_k} - a| < \epsilon.$$

Hence the claim. ///

Therefore, we have the following Criterion:
Cauchy's Criterion for convergence: A sequence $\{a_n\}$ converges if and only if for every $\epsilon > 0$, there exists N such that

$$|a_n - a_m| < \epsilon \ \forall \ m, n \geq N.$$

Example 1.0.5. *Let $\{a_n\}$be defined as $a_1 = 1, a_{n+1} = 1 + \frac{1}{a_n}$. The show that $\{a_n\}$is Cauchy. Note that $a_n > 1$ and $a_n a_{n-1} = a_{n-1} + 1 > 2$. Then*

$$|a_{n+1} - a_n| = |\frac{a_{n-1} - a_n}{a_n a_{n-1}}| \leq \frac{1}{2}|a_n - a_{n-1}| \leq \frac{1}{2^{n-1}}|a_2 - a_1|, \ \forall n \geq 2.$$

Hence

$$|a_m - a_n| \leq |a_m - a_{m-1}| + |a_{m-1} - a_{m-2}| + ... + |a_{n+1} - a_n| \leq |a_2 - a_1|\frac{\alpha^{n-1}}{1-\alpha}, \alpha = \frac{1}{2}$$

So given, $\epsilon > 0$, we can choose N such that $\frac{1}{2^{N-1}} < \frac{\epsilon}{2}$.

Indeed the following holds,

Theorem 1.0.6. *Let $\{a_n\}$ be a sequence such that $|a_{n+1} - a_n| < \alpha|a_n - a_{n-1}|$ for all $n \geq N$ for some N and $0 < \alpha < 1$. Then $\{a_n\}$ is a Cauchy sequence.*

Proof. Proof follows as in the previous example.
In the above theorem if $\alpha = 1$, then we cannot say if the sequence is Cauchy or Not. For example

Example 1.0.7. *Let $a_n = \sum_{k=1}^{n} \frac{1}{k}$. Then it is easy to see that*

$$\frac{|a_{n+1} - a_n|}{|a_n - a_{n-1}} = \frac{n}{n+1} < 1.$$

But the sequence $\{a_n\}$ diverges. Indeed, $a_{2^n} > 1 + \frac{n}{2}$, by induction.
On the other hand if we choose $a_n = \sum_{k=1}^{n} \frac{1}{k^2}$. Then

$$\frac{|a_{n+1} - a_n|}{|a_n - a_{n-1}} = \frac{n^2}{(n+1)^2} < 1.$$

But the sequence $\{a_n\}$ converges (This will be proved in the next section while studying infinite series).