Lecture 6

1 Limit superior and limit inferior

Definition 1.0.1. *Let* $\{a_n\}$*be a bounded sequence. Then limit superior of the sequence* $\{a_n\}$*, denoted by* $\limsup_{n\to\infty} a_n$*, is defined as*

$$\limsup_{n\to\infty} a_n := \inf_{k\in\mathbb{N}} \sup_{n\geq k} a_n.$$

Similarly limit inferior of the sequence $\{a_n\}$, denoted by $\liminf_{n\to\infty} a_n$, is defined as

$$\liminf_{n\to\infty} a_n := \sup_{k\in\mathbb{N}} \inf_{n\geq k} a_n.$$

Examples 1.0.2.

1. *Consider the sequence* $\{a_n\} = \{0, 1, 0, 1, ...\}$*. Then* $\beta_n = \sup\{a_m, m \geq n\} = 1$ *and* $\alpha_n = \inf\{a_m, m \geq n\} = 0$*. Therefore,* $\liminf a_n = 0, \limsup a_n = 1$*.*

2. *Consier the sequence* $a_n = \sin \frac{n\pi}{3}$*. Then by listing the elements of* $\{a_n\}$*, it is easy to see that* $\beta_n = \sup\{a_m, m \geq n\} = \frac{\sqrt{3}}{2}$*. Therefore* $\limsup a_n = \frac{\sqrt{3}}{2}$*. Similarly one can see* $\liminf a_n = -\frac{\sqrt{3}}{2}$*.*

3. *Consider the sequence* $a_n = (-1)^n \frac{n+1}{n}$*.*
Let $\beta_n = \sup\{(-1)^n \frac{n+1}{n}, (-1)^{n+1} \frac{n+2}{n+1}, ...\}$*. Then* $\beta_n = \begin{cases} \frac{n+1}{n} & n \text{ even} \\ \frac{n+2}{n+1} & n \text{ odd} \end{cases}$*. It is easy to see that* $\beta_n \to 1$*. Therefore,* $\limsup a_n = 1$*. Similarly,* $\liminf a_n = -1$*.*

4. *Consider the sequence* $\{a_n\} = \{\frac{1}{2}, \frac{2}{3}, \frac{1}{3}, \frac{3}{4}, ...\}$*. Then for large* k

$$1 \geq \sup\{a_m, m \geq k\} \geq \lim \frac{k-1}{k}$$

$$0 < \inf\{a_m, m \geq k\} \leq \lim \frac{1}{k}$$

Then by sandwich theorem, we see that $\limsup a_n = 1$ *and* $\liminf a_n = 0$*.*

Lemma 1.0.3.

(i) If $\{a_n\}$*is a bounded sequence, then* $\limsup_{n\to\infty} a_n \geq \liminf_{n\to\infty} a_n$*.*

(ii) If $\{a_n\}$*and* $\{b_n\}$ *are bounded sequences of real numbers and if* $a_n \leq b_n$ *for all* $n \in \mathbb{N}$*, then*

$$\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} b_n$$

and

$$\liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} b_n.$$

(iii) Let $\{a_n\}$and $\{b_n\}$ are bounded sequences of real numbers. Then

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty}\ a_n + \limsup_{n\to\infty}\ b_n$$

and

$$\liminf_{n\to\infty}(a_n + b_n) \geq \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n.$$

Example 1.0.4. *Consider the sequences $\{(-1)^n\}$ and $\{(-1)^{n+1}\}$. Here $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$. Also $\limsup_{n\to\infty}\ a_n = \limsup_{n\to\infty}\ b_n = 1$. But $a_n + b_n = 0$ for all $n \in \mathbb{N}$ and hence $\limsup_{n\to\infty}\ (a_n + b_n) = 0$. Thus a strict inequality may hold in (iii) the above Lemma.*

Theorem 1.0.5. *If $\{a_n\}$is a bounded sequence, then there exists subsequences $\{a_{n_k}\}$ and $\{b_{n_k}\}$ such that*

$$\limsup a_n = \lim a_{n_k} \ \ \text{and} \ \ \liminf a_n = \lim b_{n_k}.$$

Proof. Since $\{a_n\}$is bounded, $\limsup a_n = \alpha$ exists. Then from the definition, for each k large enough

$$\alpha < \sup_{n\geq k} a_n < \alpha + \frac{1}{k}$$

i.e., there exits a_{n_k} such that

$$\alpha - \frac{1}{k} < a_{n_k} < \alpha + \frac{1}{k}.$$

Therefore, $a_{n_k} \to \alpha$ as $k \to \infty$. Similarly, one can obtain b_{n_k}. ///

Theorem 1.0.6. *If there exists a subsequence $a_{n_k} \to t$. Then $t \leq s := \limsup a_n$.*

Proof. Suppose NOT. Then choose $\epsilon > 0$ such that $t - \epsilon > s$. Then we can find N such that

$$n \geq N \implies a_n < t - \epsilon$$

Therefore $|a_n - t| > \epsilon$ for all $n \geq N$. Hence such a sequence cannot have a convergent subsequence. ///

Remark 1.1. *From the above two theorems we can say that the limsup is the supremum of all limits of subsequences of a sequence.*

Remark 1.2. *In case of unbounded sequences, either* $\limsup$ *or* $\limsup$ *or both can approach* ∞. *Even in this case, one can show the existence of subsequences that approach infinity.*

Remark 1.3. *If we can find the limits of* **all** *subsequences of $\{a_n\}$. Then* $\limsup$ *is nothing but the supremum of all these limits. Similarly,* $\liminf$ *is the infimum of all these limits.*

Example 1.0.7. *Find* $\limsup$ *and* $\liminf$ *of $\{a_n\}$where $a_n = \left(1 + (-1)^n + \frac{1}{2^n}\right)^{\frac{1}{n}}$.*
The sequence $\{a_n\}$is bounded and has two convergent subsequences $\{\frac{1}{2}\}$ and $\{(2 + \frac{1}{2^n})^{\frac{1}{n}}\}$. So the two limits are $\frac{1}{2}$ and 1. Therefore, $\limsup a_n = 1$ and $\liminf a_n = \frac{1}{2}$.