Lecture 7

1 $\limsup$ and $\liminf$ ctd..

Theorem 1.0.1. *If $\{a_n\}$is a convergent sequence, then*

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} a_n = \limsup_{n\to\infty} a_n.$$

Proof. Let $L = \lim\limits_{n\to\infty} a_n$. Then given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|a_n - L| < \epsilon, \ \forall\, n \geq N.$$

Equivalently $L - \epsilon < a_n < L + \epsilon$, for all $n \geq N$. Thus, if $n \geq N$, $L + \epsilon$ is an upper bound for the set $\{a_k | k \geq N\}$. If $\alpha_k := \sup\{a_k | k \geq n\}$, then we note that $L - \epsilon < \alpha_N \leq L + \epsilon$ and $\alpha_{N+1} < L + \epsilon$, ...,$\alpha_n < L + \epsilon$ for all $n \geq N$ (As α_n is decreasing). Also $a_n > L - \epsilon$, $n \geq N \implies \alpha_n \geq L - \epsilon$, $n \geq N$. Therefore, $\lim \alpha_n = L$. Hence $\limsup a_n = L$. Similarly, one can prove the $\liminf a_n = L$.

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Theorem 1.0.2. *If $\{a_n\}$is a bounded sequence and if $\limsup\limits_{n\to\infty} a_n = \liminf\limits_{n\to\infty} a_n = L$, $L \in \mathbb{R}$, then $\{a_n\}$is a convergent sequence.*

Proof. Notice that

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} (\sup\{a_k | k \geq n\})$$

and

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} (\inf\{a_k | k \geq n\}).$$

Given that $L = \lim\limits_{n\to\infty} \sup a_n$. Thus for $\epsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that

$$|\sup\{a_n, a_{n+1}, ...\} - L| < \epsilon, \ \forall\, n \geq N_1.$$

This implies

$$a_n < L + \epsilon, \ \forall\, n \geq N_1 \tag{1.1}$$

Similarly there exists $N_2 \in \mathbb{N}$ such that

$$|\inf\{a_n, a_{n+1}, ...\} - L| < \epsilon, \ \forall\, n \geq N_2.$$

This implies

$$L - \epsilon < a_n, \ \forall\, n \geq N_2 \tag{1.2}$$

Let $N = \max\{N_1, N_2\}$. Then from (1.1)and (1.2) we get

$$|a_n - L| < \epsilon, \forall\, n \geq N.$$

Thus the sequence $\{a_n\}$converges. ///

Example 1.0.3.
$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$. *Assume that* $e = \lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{k!}$.

Solution. *Let* $a_n = \sum_{k=0}^{n}\frac{1}{k!}$ *and* $b_n = \left(1+\frac{1}{n}\right)^n$. *Now,*

$$b_n = \sum_{k=0}^{n} {}_nC_k\left(\frac{1}{n}\right)^k = 2 + \sum_{k=2}^{n}\frac{1}{k!}\prod_{i=1}^{k-1}\left(1-\frac{i}{n}\right) \leq a_n.(see\ (1.6)$$

This implies

$$\limsup_{n\to\infty}\ b_n \leq \limsup_{n\to\infty}\ a_n = e.$$

Further, if $n \geq m$, *then*

$$b_n = \left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} {}_nC_k\left(\frac{1}{n}\right)^k \geq \sum_{k=0}^{m} {}_nC_k\left(\frac{1}{n}\right)^k = 2 + \sum_{k=2}^{m}\frac{1}{k!}\prod_{i=1}^{k-1}\left(1-\frac{i}{n}\right).$$

Keeping m *fixed and letting* $n \to \infty$, *we get*

$$\liminf_{n\to\infty}\ b_n \geq \sum_{k=0}^{m}\frac{1}{k!}$$

which implies $a_n \leq \liminf_{n\to\infty}\ b_n$. *Hence*

$$e = \liminf_{n\to\infty}\ a_n \leq \liminf_{n\to\infty}\ b_n.$$

Finally we have the following more precise version of theorem 1.6.6

Theorem 1.0.4. *Let* $\{a_n\}$ *be any sequence of nonzero real numbers. Then we have*

$$\liminf\left|\frac{a_{n+1}}{a_n}\right| \leq \liminf|a_n|^{1/n} \leq \limsup|a_n|^{1/n} \leq \limsup\left|\frac{a_{n+1}}{a_n}\right|.$$

Proof. The inequality in the middle is trivial. Now we show the right end inequality. Let $L = \limsup|\frac{a_{n+1}}{a_n}|$. W.l.g assume $L < \infty$. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that

$$\left|\frac{a_{n+1}}{a_n}\right| < L+\epsilon \ \ \forall n \geq N.$$

Then for any $n > N$, we can write

$$\begin{aligned}|a_n| &= \left|\frac{a_n}{a_{n-1}}\right|\left|\frac{a_{n-1}}{a_{n-1}}\right| ... \left|\frac{a_{N+1}}{a_N}\right| |a_N| \\ &< (L+\epsilon)^{n-N}|a_N| \\ &= (L+\epsilon)^n((L+\epsilon)^{-N}|a_N|).\end{aligned}$$

Now taking $a = ((L-\epsilon)^{-N}|a_N|)$, we have, $|a_n|^{1/n} < (L+\epsilon)a^{1/n}$ for $n > N$. Since $\lim\limits_{n\to\infty} a^{1/n} = 1$, we conclude that $\limsup |a_n|^{1/n} \leq (L+\epsilon)$. Since ϵ is arbitrary, we get the result. Similarly, we can prove the first inequality.

The following example shows that the inequality can be strict in the above theorem.

Example 1.0.5. *Let* $a_n = \begin{cases} 2^n & n \text{ is even} \\ 2^{n-1} & n \text{ is odd} \end{cases}$. *Then*

$$\limsup \frac{a_{n+1}}{a_n} = 4 > \limsup a_n^{1/n} = \frac{1}{2}.$$