Lecture 8

1 Infinite Series

1.1 Definitions & convergence

Definition 1.1.1. *Let $\{a_n\}$ be a sequence of real numbers.*

a) *An expression of the form*

$$a_1 + a_2 + \ldots + a_n + \ldots$$

is called an infinite series.

b) *The number a_n is called as the n^{th} term of the series.*

c) *The sequence $\{s_n\}$, defined by $s_n = \sum_{k=1}^{n} a_k$, is called the sequence of partial sums of the series.*

d) *If the sequence of partial sums converges to a limit L, we say that the series converges and its sum is L.*

e) *If the sequence of partial sums does not converge, we say that the series diverges.*

Examples 1.1.2.

1) *If $0 < x < 1$, then $\sum_{n=0}^{\infty} x^n$ converges to $\frac{1}{1-x}$.*

Solution. *Let us consider the sequence of partial sums $\{s_n\}$, where $s_n = \sum_{k=0}^{n} x^k$. Here*

$$s_n = \sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1-x} = \frac{1}{1-x} - \frac{x^{n+1}}{1-x}, \; n \in \mathbb{N}.$$

As, $0 < x < 1$, $x^{n+1} \to 0$ as $n \to \infty$. Hence $s_n \to \frac{1}{1-x}$. Thus $\sum x^n$ converges to $\frac{1}{1-x}$. ///

2) *The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.*

Solution. *Consider the sequence of partial sums $\{s_n\}$, where $s_n = \sum_{k=1}^{n} \frac{1}{k}$. Now, let us examine the subsequence s_{2^n} of $\{s_n\}$. Here*

$$\begin{aligned} s_2 &= 1 + 1/2 = 3/2, \\ s_4 &= 1 + 1/2 + 1/3 + 1/4 > 3/2 + 1/4 + 1/4 = 2. \end{aligned}$$

Suppose $s_{2^n} > (n+2)/2$, then

$$\begin{aligned} s_{2^{n+1}} &= s_{2^n} + \sum_{k=1}^{2^n} \frac{1}{2^n + k} \\ &> \frac{n+2}{2} + \sum_{k=1}^{2^n} \frac{1}{2^{n+1}} \\ &= \frac{n+2}{2} + \frac{2^n}{2^{n+1}} = \frac{(n+1)+2}{2}. \end{aligned}$$

Thus the subsequence $\{s_{2^n}\}$ is not bounded above and as it is also increasing, it diverges. Hence the sequence diverges, i.e., the series $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges. ///

3) *(Telescopic series:) Show that the series $\sum\limits_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges to 1.*

Solution. *Consider the sequence of partial sums $\{s_n\}$. Then*

$$s_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{k} \left(\frac{1}{k} - \frac{1}{k+1} \right) = 1 - \frac{1}{n+1} \to 1.$$

Summarizing this observation, one has the following theorem on **Telescopic series**

Theorem 1.1.3. *Suppose $\{a_n\}$ is a sequence of real numbers such that $a_n \to L$. Then the series $\sum (a_n - a_{n+1})$ converges to $a_1 - L$.*

Lemma 1.1.4.

1) *If $\sum\limits_{n=1}^{\infty} a_n$ converges to L and $\sum\limits_{n=1}^{\infty} b_n$ converges to M, then the series $\sum\limits_{n=1}^{\infty} (a_n + b_n)$ converges to $L + M$.*

2) *If $\sum\limits_{n=1}^{\infty} a_n$ converges to L and if $c \in \mathbb{R}$, then the series $\sum\limits_{n=1}^{\infty} ca_n$ converges to cL.*

Lemma 1.1.5. *If $\sum\limits_{n=1}^{\infty} a_n$ converges, then $\lim\limits_{n\to\infty} a_n = 0$.*

Proof. Suppose $\sum\limits_{n=1}^{\infty} a_n = L$. Then the sequence of partial sums $\{s_n\}$ also converges to L. Now

$$a_n = s_n - s_{n-1} \to L - L = 0. \quad ///$$

Example 1.1.6. *If $x > 1$, then the series $\sum\limits_{n=1}^{\infty} x^n$ diverges.*

Solution. *Assume to the contrary that the series $\sum\limits_{n=1}^{\infty} x^n$ converges. Then the n^{th} term, i.e., $x^n \to 0$. But as $x > 1$, $x^n \geq 1$ for all $n \in \mathbb{N}$ and hence $\lim\limits_{n\to\infty} x^n \geq 1$, which is a contradiction. Hence the series $\sum\limits_{n=1}^{\infty} x^n$ diverges.* ///

As a first result we have the following comparison theorem:

Theorem 1.1.7. *Let $\{a_n\}, \{b_n\}$ be sequences of positive reals such that $a_n \leq b_n$. If $\sum b_n$ converges then $\sum a_n$ converges. Also, if $\sum a_n$ diverges then $\sum b_n$ diverges.*

Proof. Let $s_n = a_1 + a_2 + ... + a_n$ and $t_n = b_1 + b_2 + ... + b_n$ be the partial sum of $\sum a_n, \sum b_n$ respectively. Then $s_n \leq t_n$. Since $\sum b_n$ converges, we have $\{t_n\}$ converges and is bounded. Now since $\{s_n\}$ is monotonically increasing sequence that is bounded above, we get the convergence of $\{s_n\}$ and hence the convergence of $\sum a_n$.
If $\sum a_n$ diverges then $s_n \to \infty$. Then $t_n \geq s_n$ implies t_n diverges to infinity. ///

Examples 1.1.8.

$$(a) \sum \frac{1}{2^n + n} \quad (b) \sum \frac{n}{n^2 - \sin^2 n}$$

For (a), note that $2^n + n > 2^n$ and $\sum \frac{1}{2^b}$ converges. For (b) note that $n^2 - \sin^2 n < n^2$ and the series $\sum \frac{1}{n}$ diverges.