Lecture 9

1 Tests for convergence/divegence

Theorem 1.0.1. *Cauchy condensation test*
Let $\{a_n\}_1^\infty$ *be an decreasing sequence of positive numbers. Then* $\sum\limits_{n=1}^{\infty} a_n$ *converges if and only if* $\sum\limits_{n=0}^{\infty} 2^n a_{2^n}$ *converges.*

Proof. Let s_n and t_n be the sequence of partial sums of $\sum a_n$ and $\sum 2^n a_{2^n}$ respectively. Then s_n and t_n are monotonically increasing sequences. We know that such sequences converge if they are bounded from above. proof follows from the observation that

$$\begin{aligned} s_{2^n} = \sum_{k=1}^{2^n} a_n &= a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + ... + (a_{2^{n-1}+1} + ... + a_{2^n}) \\ &\geq a_1 + a_2 + 2a_4 + 4a_8 + 8a_{16} + ...2^{n-1} a_{2^n} \\ &= a_1 + \frac{1}{2} t_n. \end{aligned} \tag{1.1}$$

Therefore, if $\{s_n\}$ converges then $\{s_{2^n}\}$ converges and hence bounded from above. Now convergence of $\{t_n\}$ follows from 1.1, $\{t_n\}$.

On the other hand,

$$\begin{aligned} s_{2^n-1} &= a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + (a_8 + ...a_{15}) + (a_{2^{n-1}} + ... + a_{2^n-1}) \\ &\leq a_1 + 2a_2 + 4a_4 + 8a_8 + ... + 2^{n-1} a_{2^{n-1}} = a_1 + t_{n-1} \end{aligned}$$

So if $\{t_n\}$ converges, then $\{s_{2^n-1}\}$ converges. Now the conclusion follows from $s_n \leq s_{2^{n+1}-1}$ and the fact that $\{s_n\}$ is monotonically increasing sequence. ///

Examples 1.0.2.

1) *Consider the series* $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$, $p > 0$. *Then, we have* $\sum\limits_{n=1}^{\infty} 2^n \frac{1}{(2^n)^p} = \sum\limits_{n=1}^{\infty} \frac{1}{(2^n)^{p-1}}$ *which converges for* $p > 1$ *and diverges for* $p \leq 1$.

2) *Consider the series* $\sum\limits_{n=2}^{\infty} \frac{1}{n \log n}$. *Here* $\sum\limits_{n=2}^{\infty} 2^n \frac{1}{2^n \log 2^n} = \frac{1}{\log 2} \sum\limits_{n=2}^{\infty} \frac{1}{n}$ *which diverges. Hence the given series diverges.*

1.1 Absolute convergence

Definition 1.1.1. *a) Let* $\sum\limits_{n=1}^{\infty} a_n$ *be a series of real numbers. If* $\sum\limits_{n=1}^{\infty} |a_n|$ *converges, we say that* $\sum\limits_{n=1}^{\infty} a_n$ *converges absolutely.*

b) If $\sum\limits_{n=1}^{\infty} a_n$ *converges but* $\sum\limits_{n=1}^{\infty} |a_n|$ *diverges, we say that* $\sum\limits_{n=1}^{\infty} a_n$ *converges conditionally.*

Examples 1.1.2.

1) *The series* $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$ *converges absolutely.*

2) *The series* $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ *converges absolutely.*

Theorem 1.1.3. *If* $\sum_{n=1}^{\infty} a_n$ *converges absolutely, then* $\sum_{n=1}^{\infty} a_n$ *converges.*

Proof. Let $t_n = \sum_{k=1}^{n} |a_k|$. As the series converges absolutely, the sequence $\{t_n\}_1^\infty$ is Cauchy. Thus, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|t_m - t_n| < \epsilon \ \forall\ m, n \geq N.$$

Let $m > n$. Then

$$|s_m - s_n| = \left| \sum_{i=n+1}^{m} a_i \right| \leq \sum_{i=n+1}^{m} |a_i| = |t_m - t_n| < \epsilon.$$

Thus the sequence $\{s_n\}_1^\infty$ is Cauchy and hence converges. Thus $\sum_1^\infty a_n$ converges. ///

Theorem 1.1.4. *Let* $\sum_1^\infty a_n$ *be a series of real numbers. Let* $p_n = \max\{a_n, 0\}$ *and* $q_n = \min\{a_n, 0\}$.

a) *If* $\sum a_n$ *converges absolutely, then both* $\sum p_n$ *and* $\sum q_n$ *converges.*

b) *If* $\sum a_n$ *diverges then one of the* $\sum p_n$ *or* $\sum q_n$ *diverges.*

b) *If* $\sum a_n$ *converges conditionally then both* $\sum p_n$ *and* $\sum q_n$ *diverges.*

Proof.

a) Observe that $p_n = (a_n + |a_n|)/2$ and $q_n = (a_n - |a_n|)/2$. Thus the convergence of the two series follows from the hypothesis.

b) Proof is easy.

c) We leave to this as an exercise.

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Tests for absolute convergence

Theorem 1.1.5 (Comparison test)**.** *Let* $\sum a_n$ *be a series of real numbers. Then,* $\sum a_n$ *converges absolutely if there is an absolutely convergent series* $\sum c_n$ *with* $|a_n| \leq |c_n|$ *for all* $n \geq N, N \in \mathbb{N}$.

Examples 1.1.6.

1) *The series* $\sum_{n=1}^{\infty} \frac{7}{7n-2}$ *diverges because* $\frac{7}{7n-2} = \frac{1}{n - 2/7} \geq \frac{1}{n}$ *for all* $n \in \mathbb{N}$ *and* $\sum \frac{1}{n}$ *diverges.*

2) The series $\sum_{n=0}^{\infty} \frac{1}{n!}$ *converges because* $\frac{1}{n!} \leq \frac{1}{2^n}$ *and* $\sum_{n=0}^{\infty} \frac{1}{2^n}$ *converges.*

Theorem 1.1.7 (Limit comparison test)**.** *Let* $\{a_n\}$ *and* $\{b_n\}$ *be two sequences of positive numbers. Then*

a) if $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, $\sum a_n$ *and* $\sum b_n$ *both converge or diverge together;*

b) if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ *and* $\sum b_n$ *converges, then* $\sum a_n$ *converges.*

c) if $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ *and* $\sum b_n$ *diverges, then* $\sum a_n$ *diverges.*

Proof. (a) As $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, *for* $\epsilon = \frac{c}{2} > 0$, *there exists* $N \in \mathbb{N}$ *such that*

$$n \geq N \implies \left| \frac{a_n}{b_n} - c \right| < \frac{c}{2}.$$

Thus, for $n \geq N$,

$$\frac{-c}{2} \leq \frac{a_n}{b_n} - c \leq \frac{c}{2}$$

or equivalently

$$\frac{cb_n}{2} \leq a_n \leq \frac{3cb_n}{2}.$$

Hence the conclusion follows from the comparison test.

b) Given that $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$. *Hence for* $\epsilon = \frac{1}{2}$, *there exists* $N \in \mathbb{N}$ *such that*

$$n \geq N \implies \frac{a_n}{b_n} < \frac{1}{2}$$

or equivalently,

$$n \geq N \implies a_n \leq \frac{b_n}{2}.$$

Thus the desired conclusion follows from the comparison test.

c) Here we are given that $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$. *Hence for any real number* $M > 0$, *there exists* $N \in \mathbb{N}$ *such that*

$$n \geq N \implies \frac{a_n}{b_n} \geq M$$

or equivalently,

$$n \geq N \implies a_n \geq Mb_n.$$

Thus if $\sum |b_n|$ *diverges, then* $\sum |a_n|$ *diverges by comparison test.* ///

Examples 1.1.8.

1) Consider the series $\sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2}$*. Here* $a_n = \frac{2n+1}{(n+1)^2}$*. Let* $b_n = \frac{1}{n}$*. Then* $\frac{a_n}{b_n} = \frac{\left(\frac{2n+1}{(n+1)^2}\right)}{\frac{1}{n}} = \frac{2n^2+n}{n^2+2n+1} \to 2$ *as* $n \to \infty$*. Further,* $\sum \frac{1}{n}$ *diverges. Thus by limit comparison theorem, the given series diverges.*

2) Consider the series $\sum_{1}^{\infty} \frac{1}{2^n-1}$*. Here* $a_n = \frac{1}{2^n-1}$*. Let* $b_n = \frac{1}{2^n}$*. Then* $\frac{a_n}{b_n} = \frac{2^n}{2^n-1} \to 1$*. Further,* $\sum \frac{1}{2^n}$ *converges and hence the given series converges.*

3) Consider the series $\sum \frac{e^{-n}}{n^2}$*. Here* $a_n = \frac{e^{-n}}{n^2}$ *and* $b_n = \frac{1}{n^2}$*. Then* $\frac{a_n}{b_n} = e^{-n} \to 0$ *as* $n \to \infty$*. Further,* $\sum \frac{1}{n^2}$ *converges and hence the given series converges.*

4) Consider the series $\sum \frac{e^{-n}}{n}$*. Here* $a_n = \frac{e^{-n}}{n}$ *and* $b_n = \frac{1}{n^2}$*. Then* $\frac{a_n}{b_n} = ne^{-n} \to 0$ *as* $n \to \infty$*. Further,* $\sum \frac{1}{n^2}$ *converges and hence the given series converges.*