1) Find the critical points are their nature for $f(x,y) = y^3 + 3x^2y - 6x^2 - 6y^2 + 2$.

<u>Solution</u>

$$f_x = 6xy - 12x$$

$$f_y = 3y^2 + 3x^2 - 12y.$$

So $f_x = 0 \Rightarrow x = 0$ or $y = 2$

$f_y = 0 \Rightarrow y^2 + x^2 - 4y = 0.$

Putting $x = 0$ in $y^2 - 4y = 0$ we get $y = 0$ or $y = 4$.

Putting $y = 2$ in $y^2 + x^2 - 4y = 0$ we get $x = \pm 2$.

So possible critical points are $(0,0)$, $(0,4)$, $(2,2)$, $(-2,2)$. } 2 marks

Now $A := f_{xx} = 6y - 12$

$B := f_{xy} = f_{yx} = 6x$

$C := f_{yy} = 6y - 12$

So $AC - B^2 = (6y-12)^2 - 36x^2$

At $(0,0)$, $AC - B^2 > 0$ and $A < 0$.
So $(0,0)$ is a point of local maxima.
1 mark

At $(0,4)$, $AC - B^2 > 0$ and $A > 0$.
So $(0,4)$ is a point of local minima.
1 mark

At $(\mp 2, 2)$ $AC - B^2 < 0$.
So $(\mp 2, 2)$ are saddle points.
1 mark

2) $f(x,y) = \begin{cases} x \sin \frac{y}{x} + y \sin \frac{x}{y} & , xy \neq 0 \\ 0 & , xy = 0 \end{cases}$

Find the limit as $(x,y) \to (0,0)$ and determine continuity of f at $(0,0)$. Find the partial derivatives of f at $(0,0)$.

Solution $|f(x,y)| \leq |x| + |y|$

For $\epsilon > 0$, take $\delta = \epsilon/2$.

If $0 < \sqrt{x^2+y^2} < \delta$, then $|x| < \epsilon/2$, $|y| < \epsilon/2$.

So $|f(x,y) - 0| < \epsilon$.

So $\lim_{(x,y)\to(0,0)} f(x,y) = 0$

2 marks

Finding the limit correctly in any way 2 marks

Now $f(0,0) = 0 = \lim\limits_{(x,y)\to(0,0)} f(x,y)$.

so f is continuous at $(0,0)$.

1 mark

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = 0$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = 0$$

2 marks

3) $$f(x,y) = \begin{cases} \dfrac{\sqrt{|xy|}\,\sin xy}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$$

Test differentiability at $(0,0)$.

Solution

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = 0$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = 0$$

1 mark

$\therefore\ df(0,0) = h f_x(0,0) + k f_y(0,0) = 0.$

$$\Delta f(0,0) = f(h,k) - f(0,0)$$

$$= \frac{\sqrt{|hk|}\,\sin hk}{\sqrt{h^2+k^2}}$$

We know, f is differentiable at $(0,0)$ if and only if $\lim_{\rho\to 0} \frac{\Delta f(0,0) - df(0,0)}{\rho} = 0.$

So we need to show $\lim_{(h,k)\to(0,0)} \frac{\sqrt{|hk|}\,\sin hk}{h^2+k^2} = 0.$

1 mark

Put $h = r\cos\theta,\ k = r\sin\theta$

$$\lim_{(h,k)\to(0,0)} \frac{\sqrt{|hk|}\,\sin hk}{h^2+k^2}$$

$$= \lim_{\substack{r\to 0\\ \theta\in[0,2\pi]}} r\sqrt{|\cos\theta\sin\theta|}\,\frac{\sin(r^2\cos\theta\sin\theta)}{r^2}$$

$$= \lim_{\substack{r\to 0\\ \theta\in[0,2\pi]}} r\,|\cos\theta\sin\theta|^{3/2} \lim_{\substack{r\to 0\\ \theta\in[0,2\pi]}} \frac{\sin(r^2\cos\theta\sin\theta)}{r^2\cos\theta\sin\theta}$$

$$= 0\cdot 1 = 0$$

3 marks