Tutorial Sheet:Multivariable Calculus

1. Examine if the limits as $(x,y) \to (0,0)$ exist?

(a) $\begin{cases} \frac{x^3+y^3}{x^2-y^2} & x \neq y \\ 0 & x = y \end{cases}$ (b) $xy\left(\dfrac{x^2-y^2}{x^2+y^2}\right)$ (c) $\begin{cases} x\sin\dfrac{1}{y} + y\sin\dfrac{1}{x} & xy \neq 0 \\ 0 & xy = 0 \end{cases}$

(d) $\dfrac{\sin(xy)}{x^2+y^2}$.

2. Examine the continuity of the following functions.

(a) $\begin{cases} \dfrac{xy^3}{x^2+y^6}, & (x,y) \neq (0,0) \\ 0, & \text{otherwise.} \end{cases}$ (b) $\begin{cases} x^2+y^2, & x^2+y^2 \leq 1 \\ 0, & \text{otherwise.} \end{cases}$

(c) $\begin{cases} \dfrac{\sin^2(x-y)}{|x|+|y|}, & (x,y) \neq (0,0) \\ 0, & \text{otherwise.} \end{cases}$ (d) $\begin{cases} \dfrac{x^2y^2}{x^2y^2+(x-y)^2}, & (x,y) \neq (0,0) \\ 0, & \text{otherwise.} \end{cases}$

3. Discuss the differentiability of the following functions at $(0,0)$.

$(a) f(x,y) = \begin{cases} x\sin\dfrac{1}{x} + y\sin\dfrac{1}{y} & xy \neq 0 \\ 0 xy = 0 \end{cases}$ (b) $\begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & x^2+y^2 \neq 0 \\ 0 & x = y = 0 \end{cases}$

(c) $\begin{cases} \frac{x^6-2y^4}{x^2+y^2} & x^2+y^2 \neq 0 \\ 0 & x = 0, y = 0 \end{cases}$

4. Let $f(x,y) = \dfrac{y}{|y|}\sqrt{x^2+y^2}$, $y \neq 0$ and $f(x,0) = 0$. Show that f has all directional derivatives at $(0,0)$ but it is not differentiable at $(0,0)$.

5. Let $f(x,y) = \Big|\,|x| - |y|\,\Big| - |x| - |y|$. Is f continuous at $(0,0)$? Which directional derivatives of f exist at $(0,0)$? Is f differentiable at $(0,0)$? Give reasons.

6. Let $f(x,y) = \dfrac{1}{2}\ln(x^2+y^2) + tan^{-1}(\dfrac{y}{x})$, $P = (1,3)$. Find the direction in which $f(x,y)$ is increasing the fastest at P. Find the derivative of $f(x,y)$ in this direction.

7. A heat-seeking bug is a bug that always moves in the direction of the greatest increase in heat. Discuss the behavior of a heat seeking bug placed at a point $(2,1)$ on a metal plate heated so that the temperature at (x,y) is given by $T(x,y) = 50y^2e^{\frac{-1}{5}(x^2+y^2)}$.

8. Suppose the gradient vector of the linear function $z = f(x,y)$ is $\nabla z = (5,-12)$. If $f(9,15) = 17$, what is the value of $f(11,11)$?

9. Suppose $f(8,3) = 24$ and $f_x(8,3) = -3.4$, $f_y(8,3) = 4.2$. Estimate the values $f(9,3)$, $f(8,5)$, and $f(9,5)$. Explain how you got your estimates.

10. Find the quadratic Taylor's polynomial approximation of $e^{-x^2-2y^2}$ near $(0,0)$.