Tutorial Sheet: Definite and Improper Integrals

1. Using the definition of definite integral, determine which of the following functions defined on $[0, 1]$ are integrable

$$(a)\ f(x) = \begin{cases} 1 & x < 1 \\ 2 & x = 1 \end{cases} \quad (b)\ f(x) = \begin{cases} 1 + x & x \in \mathbb{Q} \\ 1 - x & x \notin \mathbb{Q} \end{cases} \quad (c)\ f(x) = \begin{cases} \sin x & x = \frac{1}{n}, n \in \mathbb{N} \\ \cos x & \text{otherwise} \end{cases}$$

$$(d)\ f(x) = \begin{cases} \cos \frac{\pi}{x} & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases} \quad (e)\ f(x) = \begin{cases} x[\frac{1}{x}] & 0 < x \leq 1 \\ 0 & x = 0. \end{cases}$$

where $[y]$ is integral value of y.

2. Determine if the following statements are True/false? Give a proof if it is true and give an example if it is false.
   (a) If f is integrable then f^2 and $|f|$ are also integrable.
   (b) If f, g are integrable then so is fg and $\frac{f}{g}$, $(g \neq 0)$.
   (c) If $|f|$ is integrable then f is integrable.

3. Suppose f and g are continuous functions on $[a, b]$ and $\int_a^b f = \int_a^b g$. Then show that there exists $x \in [a, b]$ such that $f(x) = g(x)$.

4. Suppose f is continuous on $[a, b]$ and $f(x) \geq 0$. Show that if $\int_a^b f(x)dx = 0$, then $f(x) = 0$.

5. Prove that

$$(a) \lim_{n\to\infty} (\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}) = log_e 2 \quad (b) \lim_{n\to\infty} \frac{1}{n}[\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \ldots + \sin\frac{n\pi}{n}] = \frac{2}{\pi}$$

6. Discuss the convergence/divergence of improper integrals of first kind

$$(a) \int_0^\infty e^{-x} cosx\ dx, \quad (b) \int_1^\infty \frac{dx}{x^2(1+e^x)} \quad (c) \int_1^\infty \frac{x+1}{\sqrt{x^3}}dx \quad (d) \int_1^\infty \frac{dx}{x^2+\sqrt{x}}$$

7. Discuss the convergence/divergence of improper integrals of second kind

$$(a) \int_1^2 \frac{\sqrt{x}}{\ln x}dx \quad (b) \int_0^1 \frac{\sin(x^2)}{\sqrt{x}}dx \quad (c) \int_1^{\pi/2} \frac{\tan x}{x^{3/2}}dx \quad (d) \int_0^3 \frac{\log x}{\sqrt{|2-x|}}dx$$

8. Discuss the convergence/divergence of improper integrals

$$(a) \int_0^\infty x^{\frac{-1}{2}} e^{x^2} dx \quad (b) \int_0^\infty \frac{x^p |\ln x|^q}{(1+x^2)^2}dx, \quad p, q > -1 \quad (c) \int_{-\infty}^\infty \frac{dx}{|x|^p(1+x^2)}dx, p > 0$$

9. Using Beta and Gamma functions, evaluate the following:

$$(a) \int_0^\infty e^{-x^2}dx \quad (b) \int_0^{\pi/2} \sqrt{\tan x}dx \quad (c) \int_0^1 x^m (\log(\frac{1}{x}))^n dx \quad (d) \int_0^{\pi/2} \sin^4\theta \cos^6\theta d\theta$$