Tutorial Sheet: Multivariable Integral Calculus

1. Evaluate the double integrals:

   (a) $\iint_R x^2 \, dA$, where R is the region bounded by $y = x^2, y = x + 2$

   (b) $\iint_R (x^2 + y^2) \, dA$, where $R : 0 \le y \le \sqrt{1 - x^2}, 0 \le x \le 1$.

   (c) $\iint_R (a^2 - x^2 - y^2) \, dA$, where R is the region $x^2 + y^2 \le a^2$.

2. Evaluate the following double integrals by changing the order of integration if needed:

   $(a)\ \int_0^3 \int_{-y}^{y} (x^2 + y^2) dx dy$ $\quad (b)\ \int_0^1 \int_2^{4-2x} dy\, dx$

   $(c)\ \int_0^\pi \int_x^\pi \frac{\sin y}{y} dy\, dx$ $\quad (d)\ \int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y} dy\, dx$

3. Find the volume of the following:

   (a) Region under the paraboloid $z = x^2 + y^2$ and above the triangle enclosed by the lines $y = x, x = 0$, and $x + y = 2$ in the xy plane.

   (b) Region bounded above by the cylinder $z = x^2$ and below by the region enclosed by the parabola $y = 2 - x^2$ and the line $y = x$ in the xy plane.

   (c) Region bounded in the first octant bounded by the coordinate planes, the cylinder $x^2+y^2 = 4$, and the plane $z + y = 3$.

   (d) Solid cut from the first octant by the cylinder $z = 12 - 3y^2$ and the plane $x + y = 2$.

   (e) Tetrahedron bounded by the planes $y = 0, z = 0, x = 0$ and $-x + y + z = 1$.

4. Use the given transformations to transform the integrals and evaluate them:

   (a) $u = 3x + 2y, v = x + 4y$ and $I = \iint_R (3x^2 + 14xy + 8y^2) \, dA$ where R is the region in the first quadrant bounded by the lines $y + \frac{3}{2}x = 1, y + \frac{3}{2}x = 3, y + \frac{1}{4}x = 0$, and $y + \frac{1}{4}x = 1$.

   (b) $u = x + 2y, v = x - y$ and $I = \int_0^{2/3} \int_y^{2-2y} (x + 2y)e^{(y-x)} dA$

   (c) $u = xy,\ v = x^2 - y^2$ and $I = \iint_R (x^2 + y^2) dA$, where R is the region bounded by $xy = 1, xy = 2, x^2 - y^2 = 1$ and $x^2 - y^2 = 2$.

5. Using appropriate transformation evaluate $\iint_R dA$, where R is the parallelogram with vertices $(1, 0), (3, 1), (2, 2)$ and $(0, 1)$.

6. Find the area of the following:

   (a) The region lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$ in the first quadrant.

   (b) The region common to the interiors of the cardioids $r = 1 + \cos\theta$ and $r = 1 - \cos\theta$.

7. Find the volume of the following solids:

   (a) Cylinder whose base lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$ and the top lies in the plane $z = x$.

   (b) Cylinder whose base is enclosed by $r^2 = 2\cos 2\theta$ and top is bounded by the sphere $z = \sqrt{2 - r^2}$.

8. Evaluate the following volume integrals:

   (a) $\iiint_D (z^2x^2 + z^2y^2)\, dV$, where $D = \{(x, y, z) \in I\!R^3, x^2 + y^2 \leq 1, -1 \leq z \leq 1\}$

   (b) $\iiint_D xyz\, dV$ where $D = \{(x, y, z) \in I\!R^3, x^2 + y^2 \leq 1,\ 0 \leq z \leq x^2 + y^2\}$

   (c) $\iiint_D e^{(x^2+y^2+z^2)^{3/2}}\, dV$ where $D = \{(x, y, z) \in I\!R^3 :\ x^2 + y^2 + z^2 \leq 1\}$

9. Find the volume of the following regions using triple integrals:

   (a) The region in the first octant bounded by the coordinate planes and the planes $x + z = 1, y + 2z = 2$.

   (b) The region in the first octant bounded by the coordinate planes, the plane $y + z = 2$, and the cylinder $x = 4 - y^2$.

   (c) The tetrahedron in the first octant bounded by the coordinate planes and the plane $x + y/2 + z/3 = 1$.

   (d) The region common to the interiors of the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$.

   (e) The region cut from the cylinder $x^2 + y^2 = 4$ by the plane $z = 0$ and the plane $x + z = 3$.

   (f) The region enclosed by $y = x^2, y = x + 2, 4z = x^2 + y^2$ and $z = x + 3$.

   (g) The region bounded above by the sphere $x^2 + y^2 + z^2 = 2$ and below by the paraboloid $z = x^2 + y^2$.

   (h) The solid bounded by the cone $z = \sqrt{x^2 + y^2}$ and paraboloid $z = x^2 + y^2$.

10. Find the mass of the solid, center of mass, moment of inertia and the radii of gyration of a solid cube in the first octant bounded by $x = 1, y = 1$ and $z = 1$. The density $\delta(x, y, z) = x+y+z+1$

11. Find the mass of solid and center of mass of a solid in the first octant bounded by the planes $y = 0, z = 0$ and by the surfaces $z = 4 - x^2$ and $x = y^2$ and its density is $\delta(x, y, z) = kxy, k$ is a constant.

12. Using the given transformation transform the given integrals and evaluate them:

    (a) $x = au, y = bv, z = cw$, and $I = \iiint_D dV$ where D is the ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

    (b) $u = x, v = xy, w = 3z$ and $I = \iiint_D (x^2y + 3xyz)\, dV$ where $D = \{(x, y, z) \in I\!R^3 : 1 \leq x \leq 2,\ 0 \leq xy \leq 2,\ 0 \leq z \leq 1\}$.

13. Find the surface area of the following:

   (a) The surface cut from the paraboloid $x^2 + y^2 - z = 0$ by the plane $z = 2$.

   (b) The surface of the region cut from the plane $x + 2y + 2z = 5$ by the cylinder whose walls are $x = y^2$ and $x = 2 - y^2$

   (c) The cap cut from the sphere $x^2 + y^2 + z^2 = 2$ by the cone $z = \sqrt{x^2 + y^2}$

   (d) The surface cut from the paraboloid $x^2 + y + z^2 = 2$ by the plane $y = 0$.

   (e) The surface cut from the cylinder $x^2 + y^2 = 4$ by the cylinder $y^2 + z^2 = 4$.

   (f) The surface cut from the cone $x^2 + y^2 = z^2$ by $x^2 + y^2 = 4x$.

   (g) The surface of the cylinder $x^2+y^2 = 4x$ which lies between the plane $z = 0$ and the cylinder $x^2 + y^2 = z^2$.

   (h) The surface area of the part of the cylinder $x^2 + y^2 = a^2$ which lies between the plane $z = 2x$ and $z = 0$.

   (i) The surface area of the part of the cylinder $x^2 + y^2 = 2x$ which lies above xy-plane and is bounded above by the paraboloid $x^2 + y^2 + z = 4$.

14. Evaluate the following line integrals:

   (a) $\int_C \overrightarrow{r}.d\overrightarrow{r}$, C being the helical path $x = \cos t,\ y = \sin t,\ z = t$ joining the points determined by $t = 0$ and $t = \pi/2$.

   (b) $\int_C \overrightarrow{v}.d\overrightarrow{r}$, where $\overrightarrow{v} = \hat{i}y + (2x)\hat{j}$, C being an arc of a circle of radius 1 centered at the origin joining (1,0) and (0,1).

   (c) $\int_C (y - z)dx + (z + x)dy + (x - y)dz$, C being a circle formed by the intersection of the sphere $x^2 + y^2 + z^2 = a^2$ and the plane $x + y + z = 0$.

   (d) $\int_C \overrightarrow{F}.d\overrightarrow{r}$, where $\overrightarrow{F} = y\hat{i} + x\hat{j} + xyz^2\hat{k}$ and C is the circle: $x^2 - 2x + y^2 = 2, z = 1$.