Tutorial Sheet: Infinite series

1. If the terms of the convergent series $\sum_{n=1}^{\infty} a_n$ are positive and forms a non-increasing sequence, then prove that $\lim_{n\to\infty} 2^n a_{2^n} = 0$.

2. If $0 \le a_n \le 1$ $(n \ge 0)$ and if $0 \le x \le 1$, then prove that $\sum_{n=1}^{\infty} a_n x^n$ converges.

3. Test the convergence of the series (1) $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ (2) $\sum_{n=2}^{\infty} \frac{1}{(\log n)^x}$ $x \in R$.

4. Determine which of the following series diverges

$$(a) \sum_{n=1}^{\infty} \frac{\log n}{n^{3/2}} \quad (b) \sum_{n=1}^{\infty} \frac{(\log n)^2}{n^{3/2}} \quad (c) \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1} \quad (d) \sum_{n=1}^{\infty} \frac{1-n}{n2^n} \quad (e) \sum_{n=1}^{\infty} \frac{1}{n\sqrt[n]{n}} \quad (f) \sum_{n=1}^{\infty} \frac{\sqrt[n]{n}}{n^2}.$$

5. Determine which of the following series converges

$$(a) \sum_{n=1}^{\infty} \frac{n^{\sqrt{2}}}{2^n} \quad (b) \sum_{n=1}^{\infty} \frac{n!}{10^n} \quad (c) \sum_{n=1}^{\infty} \left(\frac{n-2}{n}\right)^n \quad (d) \sum_{n=1}^{\infty} \frac{(\log n)^n}{n^n} \quad (e) \sum_{n=1}^{\infty} \frac{n!}{(2n+1)!}$$

6. Test the convergence of Series

$$(a) \sum_{n=0}^{\infty} (n+1+2^n) \; (b) \sum_{n=0}^{\infty} \frac{\pi^n}{3^n}, a \neq 0 \; (c) \sum_{n=0}^{\infty} \frac{1}{n!n^n} \; (d) \sum_{n=0}^{\infty} \frac{n!}{n^n} \; (e) \sum_{n=1}^{\infty} \frac{n^{n^2}}{(n+1)^{n^2}}$$

7. Test the convergence of the infinite series

$$(1) \sum_{n=0}^{\infty} \sin\left(\frac{\pi}{2^n}\right) \quad (2) \sum_{n=2}^{\infty} \frac{1}{n\ln(n^3)} \; (3) \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{1}{\sqrt{n}}\right) \; (4) \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \sin(\frac{1}{n})$$

8. Test the convergence of the infinite series

$$(1) \sum \frac{n^2 n!}{(n^3+1)n^n} \quad (2) \sum \frac{\sin \frac{n\pi}{2}}{n} \quad (c) \sum \frac{n(n!)}{(n^2+1)[(2n+1)!]}$$