Tutorial sheet: Functions

1. Determine if the following limits exist:
   $(a)\ \lim_{x\to 0}[x] \quad (b)\ \lim_{x\to 0} sgn\ (x) \quad (c)\ \lim_{x\to 0}\sin\frac{1}{x}. \quad$ (d) $\lim_{x\to 0}\sqrt{x}\sin\frac{1}{x} \quad$ (e) $\lim_{x\to 0} x\cos\frac{1}{x}$.

2. Determine if the limits exist: $\lim_{x\to 0}\frac{x-|x|}{x}$ and $\lim_{x\to\infty} x^{1+\sin x}$

3. Show that the following function f is continuous only at $x = 1/2$.

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 1-x & \text{if } x \text{ is irrational} \end{cases}$$

4. Determine which of the following functions are uniformly continuous in the interval mentioned:
   $(a)\ e^{x^2}\sin(x^2)$ in $(0,1)$ $\quad (b)\ |\sin x|$ in $[0,\infty)$ $\quad (c)\ \sqrt{x}\sin x$ in $\mathbb{R}$ $\quad$ (d) $\sin(x^2)$ in $\mathbb{R}$

5. Determine if the following functions are differentiable at 0. Find $f'(0)$ if exists
   $(a)\ \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases} \quad (b)\ e^{-|x|},\quad x \in \mathbb{R} \quad (c)\ \begin{cases} x\cos\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

6. Determine if f' is continuous at 0 for the following functions:
   $(a)\ \begin{cases} x^3\sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases} \quad (b)\ \begin{cases} x^2\cos\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases} \quad (c)\ \begin{cases} x^2\ln\frac{1}{|x|} & x \neq 0 \\ 0 & x = 0 \end{cases}$

7. Let f be differentiable on $I\!R$ and $\sup_{I\!R}|f'(x)| < 1$. Select $s_0 \in I\!R$ and define $s_n = f(s_{n-1})$. Prove that $\{s_n\}$ is a convergent sequence.

8. Let f be differentiable on $I\!R$ and $|f(x) - f(y)| \leq (x-y)^2$. Then show that f is constant.

9. Evaluate the following limits

$$(a)\ \lim_{x\to 0}\frac{e^x - (1+x)}{x^2} \quad (b)\ \lim_{t\to 0}\frac{1-\cos t - (t^2/2)}{t^4} \quad (c)\ \lim_{x\to\infty} x^2(e^{-1/x^2} - 1)$$

10. Find the approximation of $\sin x$ when error is of magnitude no greater than 5×10^{-4} and $|x| < 3/10$.

11. Estimate the error in the approximation of $\sin hx = x + (x^3/3!)$ when $|x| < 0.5$.

12. Find the radius of convergence of Power Series

$$(a)\sum_{n=0}^{\infty}(n+1+2^n)x^n \ (b)\ \sum_{n=0}^{\infty}\frac{x^{2n}}{a^n}, a\neq 0 \ (c)\sum_{n=0}^{\infty}\frac{x^n}{n!n^n} \ (d)\ \sum_{n=0}^{\infty}\frac{n!x^n}{n^n} \ (e)\ \sum_{n=1}^{\infty}\frac{n^{n^2}}{(n+1)^{n^2}}(x-1)^n$$

13. Write the Taylor's series around 0 and find the radius of convergence of

$$(1)\ \frac{1}{1+x} \quad (2)\ \sinh x \quad (3)\ e^x\sinh x \quad (4)\ x\sin x$$

14. Obtain the Talylor's series around 0 using term by term differentiation/integration and calculate the radius of convergence of series. Is this the maximal interval of validity of the series?

$$(a)\ \tan^{-1}(x), \quad (b)\ \sin^{-1}x, \quad (c)\ \sinh^{-1}x \quad (d)\ \frac{1}{(1+x^2)^2}$$